Lesson 3
Factoring to Solve Quadratic Equations

Objectives

In this lesson you will:

- ✓ Identify monomials, binomials, and trinomials, and recognize that these are all polynomials.
- ✓ Factor a trinomial of the form $x^2 + bx + \mathrm{c}$ or $ax^2 + bx + c$.
- ✓ Recognize and factor a difference of two squares or a perfect square trinomial.
- ✓ Check to see if the terms of a given polynomial have a common monomial factor.
- ✓ Solve quadratic equations by factoring.
- ✓ Solve realistic problems using quadratic equations.

Monomials, Binomials, and Trinomials

A **polynomial function** is a function of the form $f(x) = a_n x^n + a_{n-1} x^{n-1} + \ldots\ldots + a_1 x^1 + a_0$ where "n" is a nonnegative number. A polynomial with only one term is called a **monomial.** A polynomial with two terms is a **binomial**. A polynomial with three terms is called a **trinomial**.

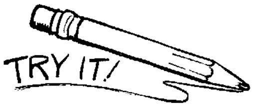
Example 1

Observe the table on the next page. If the function is a polynomial function, classify it by the number of terms. If it is not a polynomial function, state why.

Function	Polynomial Function	Classification of Polynomial by Number of Terms
$f(x) = x^2 + 2x + 3$		
$f(x) = x^2 + 2$		
$f(x) = x^{-3} + 3x + 2$		
$f(x) = 4$		
$f(x) = \sqrt{x} - 2x^{-4} + 3$		
$f(x) = 3x$		

Solution

Function	Polynomial Function	Classification of Polynomial by Number of Terms
$f(x) = x^2 + 2x + 3$	Yes, it is a polynomial function.	Trinomial
$f(x) = x^2 + 2$	Yes, it is a polynomial function.	Binomial
$f(x) = x^{-3} + 3x + 2$	No, it is not a polynomial function because x^{-3} has a negative power.	
$f(x) = 4$	Yes, it is a polynomial function.	Monomial
$f(x) = \sqrt{x} - 2x^{-4} + 3$	No, it is not a polynomial function because $2x^{-4}$ has a negative power.	
$f(x) = 3x$	Yes, it is a polynomial function.	Monomial



1. If the function is a polynomial function, classify it by the number of terms. If it is not a polynomial function, state why.

Function	Polynomial Function	Classification
a. $f(x) = 2x^2 + 4x + 3$		
b. $f(x) = x^2 - 3x + 4$		
c. $f(x) = 4x$		
d. $f(x) = x^2 + 2x$		
e. $f(x) = \sqrt{20x} - 2x^{-6} + 4$		
f. $f(x) = x + 3$		
g. $f(x) = 7$		
h. $f(x) = x^{-5} + 4x + 3$		

Factoring a Quadratic Trinomial when the Coefficient *"a"* = 1

A **quadratic trinomial** is a polynomial that can be written in the form $ax^2 + bx + c$. A quadratic trinomial can be written as the product of two binomials. Writing a quadratic trinomial as the product of two binomials is called **factoring**. The expression $x^2 + 11x + 18$ is a quadratic trinomial. It can be written as the product of two binomials $(x + 9)(x + 2)$. Notice that the sum of 9 and 2 is 11, which is the coefficient of the middle term. Also, the product of 9 and 2 is 18, which is the last term in the trinomial.

Note: To factor a quadratic trinomial of the form $x^2 + bx + c$, find integers "p" and "q" such that: $x^2 + bx + c = (x + p)(x + q) = x^2 + (p + q)x + pq$. The sum of "p" and "q" is "b" and the product of "p" and "q" is "c".

Example 2

Factor the quadratic trinomial $x^2 + 12x + 36$.

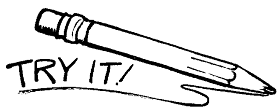
Solution

Let $x^2 + 12x + 36 = (x + p)(x + q)$. Find two numbers "$p$" and "$q$" such that their sum is 12 and their product is 36.

Factors of 36	**Sum of factors**
1, 36	1 + 36 = 37
–1, –36	–1 + –36 = –37
2, 18	2 + 18 = 20
–2, –18	–2 + (–18) = –20
3, 12	3 + 12 = 15
–3, –12	–3 + (–12) = –15
4, 9	4 + 9 = 13
–4, –9	–4 + (–9) = –13
6, 6	6 + 6 = 12
–6, –6	–6 + (–6) = –12

The table shows that when $p = 6$ and $q = 6$, the sum is 12.

So, $x^2 + 12x + 36 = (x + 6)(x + 6)$. Check your answer by multiplying the two binomials: $(x + 6)(x + 6) = x^2 + 6x + 6x + 36 = x^2 + 12x + 36$



Factor each quadratic trinomial below.

2. $x^2 + 10x + 25$

3. $x^2 - 4x + 3$

4. $x^2 + 9x + 14$

5. $x^2 - 7x + 10$

Factoring a Quadratic Trinomial when the Coefficient "a" $\neq 1$

The expression $2x^2 + 10x + 12$ is a quadratic trinomial with the coefficient *"a"* equal to two. It can be written as the product of two binomials $(2x + 4)(x + 3)$. Notice that the product of the coefficients of x (2 and 1) is 2. This is the coefficient for x^2 in the trinomial. The product of 4 and 3 is 12. This is the last term in the trinomial. The sum of 2•3 and 1•4 is 6+4 or 10, the coefficient of the middle term.

Note: To factor a quadratic trinomial $ax^2 + bx + c$ when $a \neq 1$, find integers m, n, p, and q such that: $ax^2 + bx + c = (mx + p)(nx + q) = mnx^2 + (mq + np)x + pq$. The product mn *represents the coefficient of the first term (a). The product pq represents the third term in the trinomial. The sum of mq and np represents the coefficient of the middle term (b).*

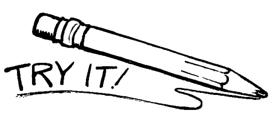
Example 3

Factor the quadratic trinomial $5x^2 - 7x + 2$.

Solution

Pairs of Binomials	Sum of *nq* and *pq* (Goal = $-7x$)
$(5x - 1)(x - 2)$	$-10x + (-1x) = -11x$
$(5x - 2)(x - 1)$	$-5x + (-2x) = -7x$

Thus the correct factorization for $5x^2 - 7x + 2$ is $(5x - 2)(x - 1)$. Check your answer by multiplying the two binomials: $(5x - 2)(x - 1) = 5x^2 - 5x - 2x + 2 = 5x^2 - 7x + 2$



Factor each of the following quadratic trinomials.

6. $2x^2 + 7x + 3$

7. $10x^2 - 16x + 6$

8. $12x^2 - 25x - 7$

Factoring Special Quadratic Trinomials

Factoring a quadratic trinomial involves guessing and checking different solutions. However, there are two special quadratic trinomials that are easier to factor. They follow a special pattern called **perfect square trinomials.** Another common quadratic expression that is easy to factor is a binomial that is the **difference of two squares.**

Difference of two squares:

$$a^2 - b^2 = (a + b)(a - b)$$

Perfect square trinomials:

$$a^2 - 2ab + b^2 = (a - b)^2$$

$$a^2 + 2ab + b^2 = (a + b)^2$$

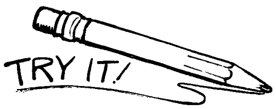
Example 4

Factor the quadratic expression and identify its special pattern.

Quadratic Expression	Rewrite Terms as $a^2 - b^2$, $a^2 - 2ab + b^2$, or $a^2 + 2ab + b^2$	Product of Two Binomials	Special Pattern
$f(x) = 9x^2 - 25$			
$f(x) = x^2 + 4x + 4$			
$f(x) = x^2 + 12x + 36$			
$f(x) = 4x^2 - 8x + 4$			

Solution

Quadratic Expression	Rewrite Terms as $a^2 - b^2$, $a^2 - 2ab + b^2$, or $a^2 + 2ab + b^2$	Product of Two Binomials	Special Pattern
$f(x) = 9x^2 - 25$	$(3x)^2 - 5^2$	$(3x - 5)(3x + 5)$	Difference of two squares
$f(x) = x^2 + 4x + 4$	$x^2 + 2(1 \bullet 2)(x) + 2^2$	$(x + 2)^2$	Perfect square trinomial
$f(x) = x^2 + 12x + 36$	$x^2 + 2(1 \bullet 6)x + 6^2$	$(x + 6)^2$	Perfect square trinomial
$f(x) = x^2 - 4x + 4$	$x^2 - 2(1 \bullet 2)x + 2^2$	$(x - 2)^2$	Perfect square trinomial



Factor each of the following quadratic expressions and identify the special pattern for each.

9. $f(x) = x^2 + 18x + 81$

10. $f(x) = 9x^2 + 24x + 16$

11. $f(x) = 100x^2 - 49$

12. $f(x) = 25x^2 - 60x + 36$

Common Monomial Factor

Recall that a **monomial** is an expression that has only one term. As a first step to factoring, determine if each term in the expression has a common monomial factor. If each term has the same monomial factor, this monomial needs to be factored out first.

Example 5

Factor the quadratic expression $f(x) = 5x^2 - 45$.

Solution

$f(x) = 5x^2 - 45 = 5(x^2 - 9) = 5(x - 3)(x + 3)$

Example 6

Factor the quadratic trinomial $f(x) = 5x^2 + 5x - 10$.

Solution

$f(x) = 5x^2 + 5x - 10 = 5(x^2 + x - 2) = 5(x + 2)(x - 1)$

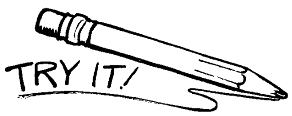
Example 7

Factor the quadratic trinomial $f(x) = -x^2 + 2x - 1$.

Solution

If we factor out –1, we will have a perfect square trinomial.

$f(x) = -x^2 + 2x - 1 = -1(x^2 - 2x + 1) = -1(x - 1)^2$



Factor each quadratic expression below.

13. $f(x) = 8x^2 - 28x - 60$

14. $f(x) = 6x^2 - 600$

15. $f(x) = 4x^2 - 8x - 32$

Solving a Quadratic Equation

When a quadratic function is written in standard form $(\boldsymbol{ax^2 + bx + c})$ and set equal to zero, it can be solved by factoring the trinomial into two binomials, setting each binomial equal to zero, and then solving for x.

Example 8

Solve the equation $x^2 + 13x + 36 = 0$.

Solution

Step 1: Write the original equation $x^2 + 13x + 36 = 0$.

Step 2: Factor.

$x^2 + 13x + 36 = (x + 4)(x + 9) = 0$

Step 3: Set each binomial equal to zero.

$(x + 4) = 0$ or $(x + 9) = 0$

Step 4: Solve for x in each binomial.

$x + 4 = 0$ or $x + 9 = 0$

$x = –4$ or $x = –9$

The solutions are $x = –4$ and $x = –9$. Check the solutions in the original equation.

Step 1: Write the original function $x^2 + 13x + 36 = 0$.

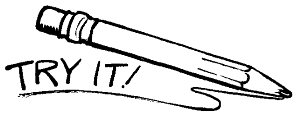
Step 2: Substitute $x = –4$ into the equation.

$(–4)^2 + 13(–4) + 36 = 16 + (–52) + 36 = 0$

Step 3: Substitute $x = –9$ into the equation.

$(–9)^2 + 13(–9) + 36 = 81 + (–117) + 36 = 0$

Since both equations equal zero, $x = –4$ and $x = –9$ are solutions.



Solve each quadratic equation below.

16. $4x^2 + 12x - 7 = 0$

17. $81x^2 - 16 = 0$

18. $18x^2 - 2 = 0$

19. $2x^2 + 7x + 3 = 0$

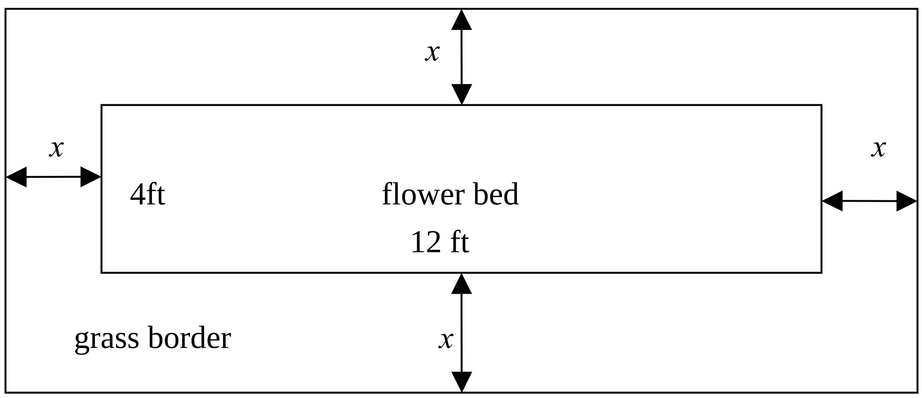
20. $5x^2 + 5x - 10 = 0$

Example 9

A rectangular flower bed is 4 feet wide and 12 feet long. Around the flowerbed is a grass border that has a uniform width (the same all the way around) and an area four times the area of the flower bed. Find the width of the border.

Solution

Step 1: Draw a picture and define the variables. Let x = the width of the grass border.



Step 2: Write a sentence that will help you find the answer to the problem.

The area of the large rectangle is the area of the flower bed plus four times the area of the grass border.

Step 3: Write mathematical expressions for Step 2.

Area of the large rectangle: $(2x + 4)(2x + 12)$

Area of flower bed: $4 \bullet 12$

Area of the grass border: $4(4 \bullet 12)$

Step 4: Write a mathematical equation for the sentence you wrote in Step 2 using the mathematical expressions you identified in Step 3.

$(2x + 4)(2x + 12) = 4 \bullet 12 + 4(4 \bullet 12)$

Step 5: FOIL the left side of the equation and simplify the right side of the equation.

$(2x + 4)(2x + 12) = 240$

$4x^2 + 24x + 8x + 48 = 240$

$4x^2 + 32x + 48 = 240$

Step 6: Subtract 240 from each side and add like terms.

$4x^2 + 32x + 48 – 240 = 240 – 240$

$4x^2 + 32x – 192 = 0$

Think Back
FOIL means
Multiply **F**irst Terms
Multiply **O**utside Terms
Multiply **I**nside Terms
Multiply **L**ast Terms

Step 7: Factor out the monomial 4.

$4(x^2 + 8x – 48) = 0$

Step 8: Factor the quadratic trinomial into two binomials.

$4(x + 12)(x – 4) = 0$

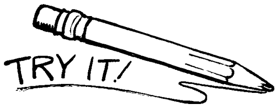
Step 9: Solve for *x*.

$x + 12 = 0$ or $x – 4 = 0$

$x = –12$ or $x = 4$

Step 10: Determine which solution is correct.

Since *x* represents a distance and distance cannot be negative, $x = –12$ is not a solution. The only possible solution is $x = 4$. Therefore the uniform width of the grass border around the flower bed is 4 feet.



21. A rectangular flower bed is 10 feet wide and 12 feet long. There is a grass border around the flower bed whose width is uniform and whose area is $2\frac{2}{3}$ times the area of the flower bed. Find the width of the border.

Example 10

The finance director of a large magazine company wants to increase the number of subscribers to her company's magazine. When the subscription price is $15 per year, the company has 20,000 subscribers. For each $1 increase in price, the company expects to lose 1,000 subscribers. How much should the company charge to maximize the annual revenue? What is the maximum revenue?

Solution

Step 1: Write a sentence that will help you solve the problem.

The company's revenue is the number of subscribers times the subscription price.

Step 2: Identify the mathematical parts of step 1 and define the variables.

Let x = the number of $1 price increases and let R = the annual revenue.

Number of subscribers = $20{,}000 - 1000x$

Subscription price = $15 + x$

Step 3: Write a mathematical expression for the sentence you wrote in Step 1.

$R = (20{,}000 - 1000x)(15 + x)$

Step 4: Let $R = 0$ and solve for x.

$(20{,}000 - 1000x)(15 + x) = 0$

$20{,}000 - 1000x = 0$ or $15 + x = 0$

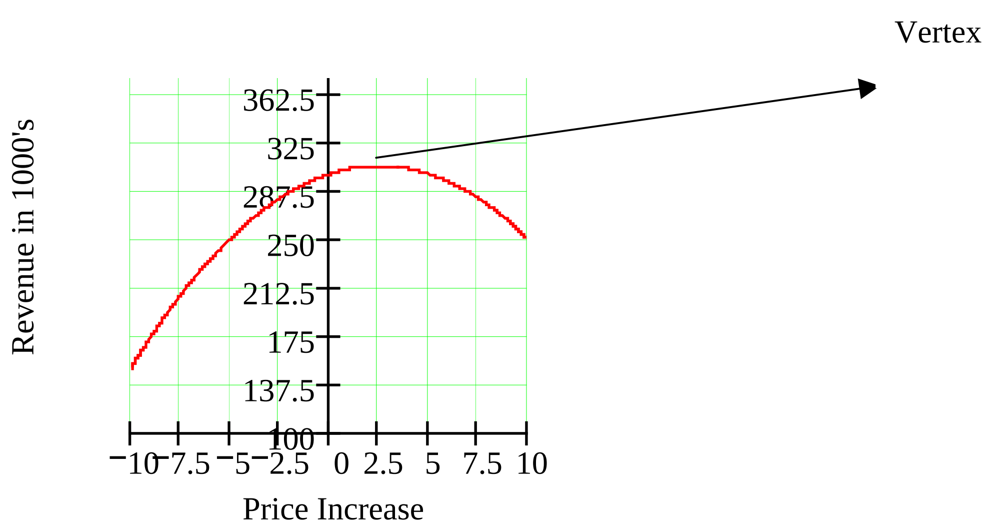
$x = 20$ or $x = -15$

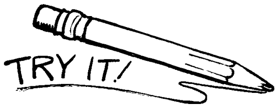
Step 5: To find the value of x that maximizes R, find the average of the x-values found in step 4.

$\frac{20 + -15}{2} = 2.5$

Step 6: To maximize revenue, charge \$15 + \$2.50 = \$17.50. Your maximum revenue is $R = (20{,}000 - 1000 \cdot 2.5)(2.5 + 15) = \$306{,}250$.

The graph of the function $R = (20{,}000 - 1000x)(15 + x)$ shows that the maximum revenue is the vertex.





22. The finance director of a large magazine company wants to increase the number of subscribers to her company's magazine. When the subscription price is $40 per year, the company has 25,000 subscribers. For each $1 increase in price, the company expects to lose 500 subscribers. How much should the company charge to maximize the annual revenue? What is the maximum revenue?

Review

1. Locate and highlight the vocabulary words and their meaning in this lesson.

2. Write one new thing that you learned from this lesson or one question that you would like to ask your mentor.

Practice Problems

Unit 3 Lesson 3

Directions: Write your answers in your math notebook. Label this exercise Unit 3 – Lesson 3 Set A, Set B, Set C, and Set D.

Set A

1. Observe the table below. If the function is a polynomial function, classify it by the number of terms. If it is not a polynomial function, state why.

Function	Polynomial Function	Classification of Polynomial by Number of Terms
$f(x) = x^2 + 3x + 4$		
$f(x) = 2x^2 + 3$		
$f(x) = 3x^{-3} + 3x + 4$		
$f(x) = 5$		
$f(x) = 15x^2 - 20$		
$f(x) = \sqrt{2x} - 2x^{-5} + 3$		
$f(x) = 3x + 4$		

Set B

Factor each quadratic trinomial below.

1. $2x^2 + 20x + 50$

2. $x^2 – 6x – 7$

3. $x^2 + 8x + 15$

4. $x^2 – 7x – 18$

5. $6x^2 + 21x + 9$

6. $20x^2 – 38x + 12$

7. $8\underline{x}^2 + 28x + 24$

Set C

Factor each quadratic expression below and identify its special pattern.

1. $f(x) = x^2 + 16x + 64$

2. $f(x) = 25x^2 + 40x + 16$

3. $f(x) = 81x^2 – 16$

4. $f(x) = 4x^2 – 12x + 9$

Solve each quadratic equation below.

5. $x^2 + 12x + 27 = 0$

6. $100x^2 – 49 = 0$

7. $18x^2 – 3x – 3 = 0$

8. $2x^2 – 10x – 12 = 0$

9. $15x^2 + 15x – 30 = 0$

Set D

1. A rectangular flower bed is 8 feet wide and 12 feet long. Around the flower bed there is a grass border whose width is uniform and whose area is 4 times the area of the flower bed. Find the width of the border.

2. The finance director of a large magazine company wants to increase the number of subscribers to her company's magazine. When the subscription price is \$35 per year, the company has 15,000 subscribers. For each \$1 increase in price, the company can expect to lose 250 subscribers. How much should the company charge to maximize their annual revenue? What is the maximum revenue?

ANSWERS TO TRY IT

1. a. The function is a polynomial. It is a trinomial.

b. The function is a polynomial. It is a trinomial.

c. The function is a polynomial. It is a monomial.

d. The function is a polynomial. It is a binomial.

e. The function is not a polynomial because x^{-6} has a negative power.

f. The function is a polynomial. It is a binomial.

g. The function is a polynomial. It is a monomial.

h. The function is not a polynomial because x^{-5} has a negative power.

2. $x^2 + 10x + 25 = (x + 5)(x + 5)$

3. $x^2 - 4x + 3 = (x - 3)(x - 1)$

4. $x^2 + 9x + 14 = (x + 7)(x + 2)$

5. $x^2 - 7x + 10 = (x - 5)(x - 2)$

6. $2x^2 + 7x + 3 = (2x + 1)(x + 3)$

7. $10x^2 - 16x + 6 = (5x - 3)(2x - 2)$

8. $12x^2 - 25x - 7 = (3x - 7)(4x + 1)$

9. $x^2 + 18x + 81 = (x + 9)^2$; Perfect square trinomial

10. $9x^2 + 24x + 16 = (3x + 4)^2$; Perfect square trinomial

11. $100x^2 - 49 = (10x - 7)(10x + 7)$; Difference of two squares

12. $25x^2 - 60x + 36 = (5x - 6)^2$; Perfect square trinomial

13. $8x^2 - 28x - 60 = 4(2x^2 - 7x - 15) = 4(2x + 3)(x - 5)$

14. $6x^2 - 600 = 6(x^2 - 100) = 6(x - 10)(x + 10)$

15. $4x^2 - 8x - 32 = 4(x^2 - 2x - 8) = 4(x - 4)(x + 2)$

16. $4x^2 + 12x - 7 = (2x - 1)(2x + 7) = 0$; $x = \frac{1}{2}$ or $x = -\frac{7}{2}$

17. $81x^2 - 16 = (9x - 4)(9x + 4) = 0$; $x = \frac{4}{9}$ or $x = -\frac{4}{9}$

18. $18x^2 - 2 = 2(9x^2 - 1) = 2(3x - 1)(3x + 1) = 0$; $x = \frac{1}{3}$ or $x = -\frac{1}{3}$

19. $2x^2 + 7x + 3 = (2x + 1)(x + 3) = 0$; $x = -\frac{1}{2}$ or $x = -3$

20. $5x^2 + 5x - 10 = 5(x^2 + x - 2) = 5(x + 2)(x - 1) = 0$; $x = -2$ or $x = 1$

21. $(2x + 10)(2x + 12) = 10 \cdot 12 + 2\frac{2}{3}(10 \cdot 12)$

$4x^2 + 44x + 120 = 120 + 320$

$4x^2 + 44x - 320 = 0$

$4(x^2 + 11x - 80) = 0$

$4(x - 5)(x + 16) = 0$

$x = 5$ or $x = -16$

Since x represents distance and distance can't be negative, $x = -16$ is not a solution. The only possible solution is $x = 5$. Therefore the uniform width of the grass border around the flower bed is 5 feet.

22. $R = (25{,}000 - 500x)(40 + x)$

When $R = 0$, $x = 50$ or - 40.

The average value of the x-values is 5. To maximize revenue, charge \$40 + \$5 = \$45. The maximum revenue is $R = (25{,}000 - 500 \bullet 45)(40 + 45) =$ \$212,500.

End of Lesson 3