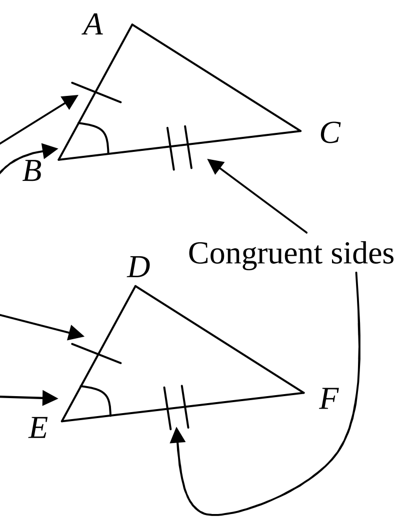
Propositions

- **Proposition 1: Theorem 2.1**

If two sides and the included angle of one triangle are congruent to two sides and the included angle of another triangle, the triangles are congruent. s.a.s. = s.a.s.

With this theorem, we can prove these two triangles are congruent if we can show the following:

- Two corresponding sides are congruent
  $\overline{AB} \cong \overline{DE}$, and $\overline{BC} \cong \overline{EF}$
  (denoted by the hash marks)
- The angles in between the sides are congruent (denoted by the arcs)

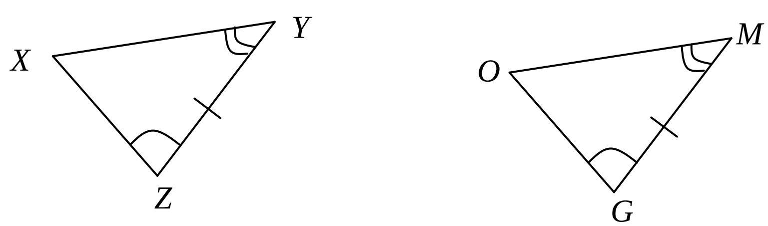


This theorem is often called "<u>S</u>ide <u>A</u>ngle <u>S</u>ide," or SAS.

- **Proposition 2: Theorem 2.2**

If two angles and the included side of one triangle are congruent to two angles and the included side of another triangle, the triangles are congruent. a.s.a. = a.s.a.

The following triangles were proven to be congruent by this theorem, ASA.



$\angle XYZ \cong \angle OMG$ (A)

$\overline{YZ} \cong \overline{MG}$ (S)

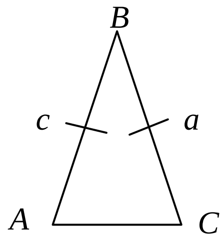
$\angle YZX \cong \angle MGO$ (A)

You need to show that two sets of corresponding angles are congruent, and as well as the side in between each set of angles.

- **Proposition 3: Theorem 2.3**

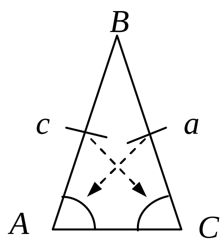
  If a triangle is isosceles, the angles opposite the equal sides are equal.

  Given an isosceles triangle (a triangle with two equal sides),



$\overline{a} \cong \overline{c}$

this theorem tells us that the angles opposite the congruent sides are congruent.

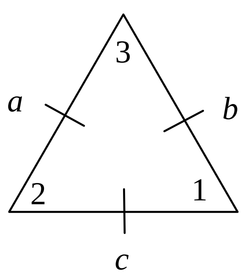


$\angle A \cong \angle C$.

C: ***An equilateral triangle is equiangular***

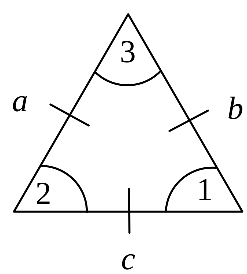
Using the previous theorem, we find that in an equilateral triangle (a triangle with three equal sides), all the angles are congruent.

Given this,



$\overline{a} \cong \overline{b} \cong \overline{c}$

you conclude this.



$\angle 1 \cong \angle 2 \cong \angle 3$

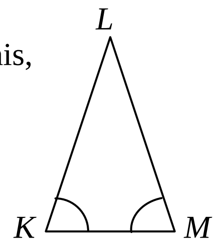
- **Proposition 4: Theorem 2.4**

  If two angles of a triangle are equal, the sides opposite the equal angles are equal.

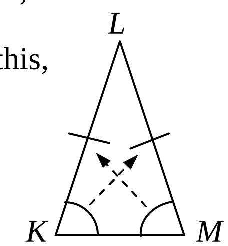
  This is the backwards version, or the converse, of Theorem 2.3.

Given this,

$\angle K \cong \angle M$



conclude this,

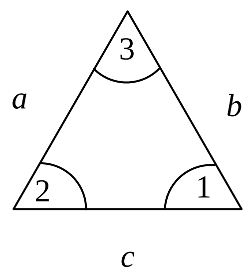


$\overline{KL} \cong \overline{ML}$

C: ***If a triangle is equiangular, it is also equilateral.***

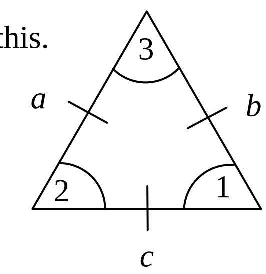
Given a triangle with three congruent angles, you conclude all three sides are also congruent.

Given this,



$\angle 1 \cong \angle 2 \cong \angle 3$

you conclude this.



$\overline{a} \cong \overline{b} \cong \overline{c}$

- **Proposition 5: Theorem 2.5**

If three sides of one triangle are congruent to three sides of another triangle, the triangles are congruent. s.s.s = s.s.s.

If you can show that all three corresponding sides of two triangles congruent, you can conclude that the triangles are congruent.

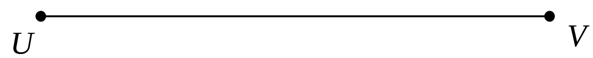
We have $\overline{AB} \cong \overline{DE}$, $\overline{BC} \cong \overline{EF}$, and $\overline{AC} \cong \overline{DF}$, so we conclude that

$$\Delta ABC \cong \Delta DEF$$

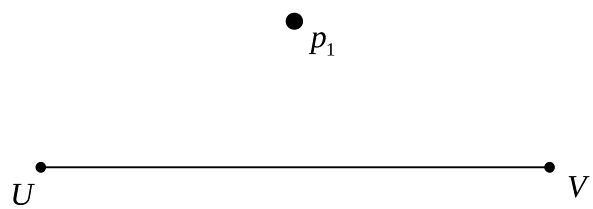
- **Proposition 6: Theorem 2.6**

  If two points are each equidistant from the ends of a segment, they determine the perpendicular bisector of the segment.

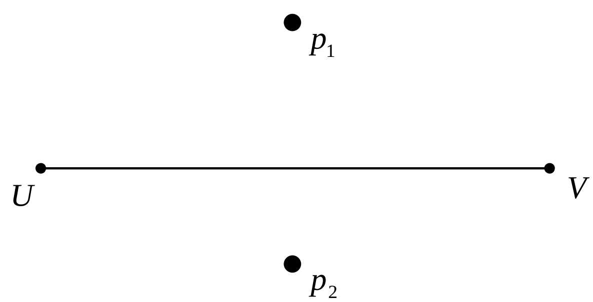
Start with a line segment, $\overline{UV}$.



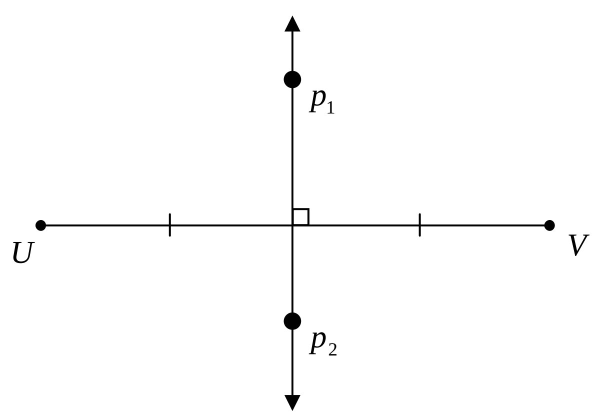
Take a point, p_1, that is the same distance from *U* as it is from *V*.



Take another point, p_2, that is also the same distance from *U* and it is from *V*.



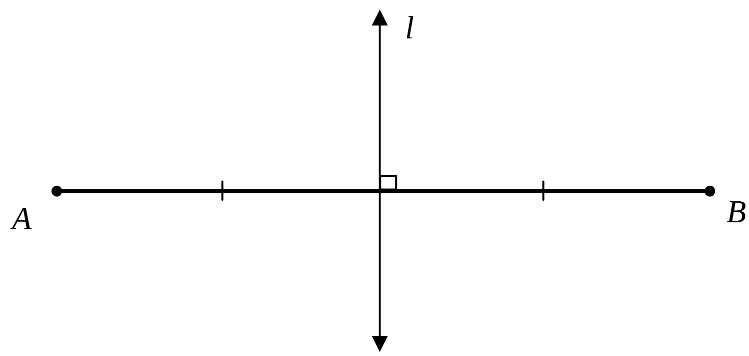
The line that goes through p_1 and p_2 will form right angles (be perpendicular), and will cross $\overline{UV}$ in its exact middle (bisector).



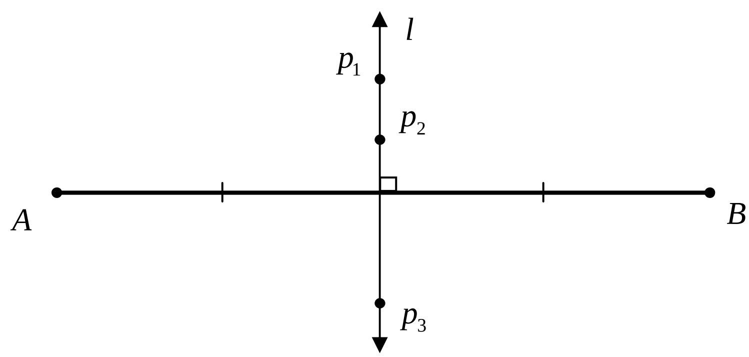
- **Proposition 7: Theorem 2.7**

  Any point on the perpendicular bisector of a segment is equidistant from the ends of the segment.

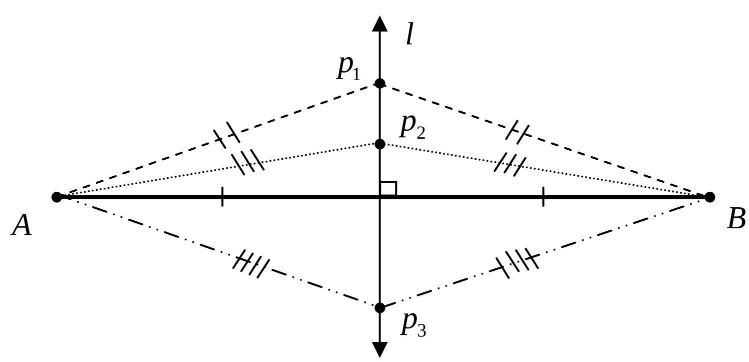
Starting with a line segment, $\overline{AB}$, and its perpendicular bisector, line *l*,



we take some points that are on line *l*.



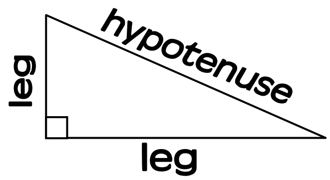
We know that each point is the same distance from *A* as it is from *B*.



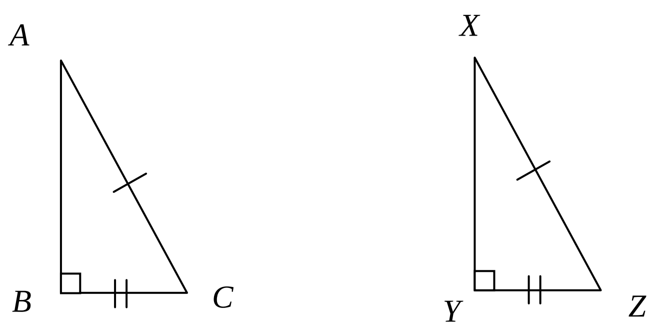
Proposition 8: Theorem 2.8

Two right triangles are congruent if the hypotenuse and a leg of one are equal, respectively, to the hypotenuse and a leg of the other. h.l. = h.l.

Recall the parts of a right triangle:



This theorem states that if a leg and hypotenuse of one triangle are congruent to a leg and hypotenuse of another, the triangles are congruent. Observe:



These are right triangles ΔABC and ΔXYZ.

Their hypotenuses are congruent $\left(\overline{AC} \cong \overline{XZ}\right)$.

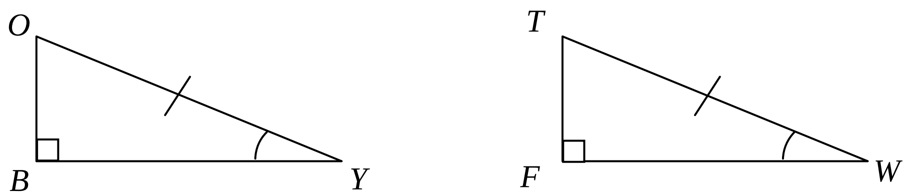
We also see that each has a congruent leg $\left(\overline{BC} \cong \overline{YZ}\right)$.

Therefore, the triangles are congruent $\left(\Delta ABC \cong \Delta XYZ\right)$.

- **Proposition 9: Theorem 2.9**

  Two right triangles are congruent if the hypotenuse and an adjacent angle of one are congruent to the hypotenuse and an adjacent angle of another. h.a. = h.a.

  Here is an example.



  These are right triangles ΔOBY, and ΔTFW.

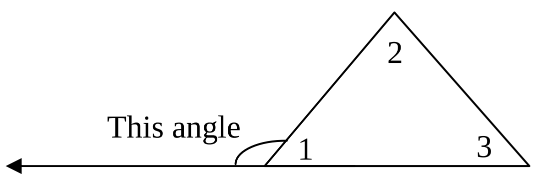
  Their hypotenuses are congruent $\left(\overline{OY} \cong \overline{TW}\right)$.

  The angles adjacent (next to) each hypotenuse are also congruent $\left(\angle OYB \cong \angle TWF\right)$.

  Therefore, the triangles are congruent $\left(\Delta OBY \cong \Delta TFW\right)$.

- **Proposition 10: Theorem 3.1**

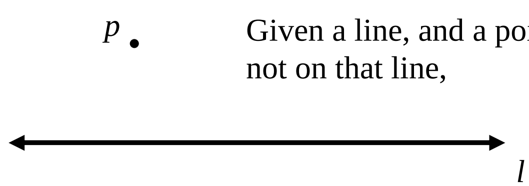
  An exterior angle of a triangle is greater than either opposite interior angle.



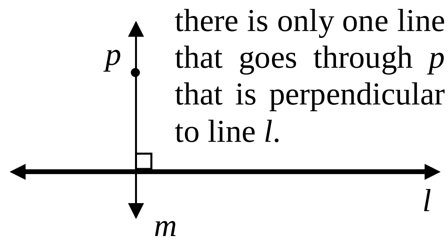
  is always bigger than angle 2 or angle 3.

- **Proposition 11: Theorem 3.2**

  One and only one perpendicular can be drawn to a given line through a given point.



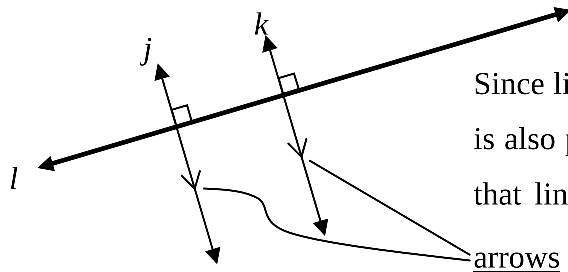
  Given a line, and a point not on that line,



  there is only one line that goes through *p* that is perpendicular to line *l*.

- **Proposition 12: Theorem 3.3**

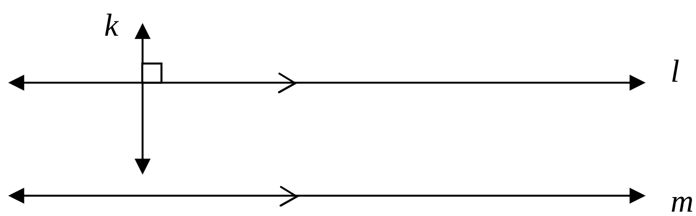
  Two lines in the same plane, perpendicular to the same line, are parallel.



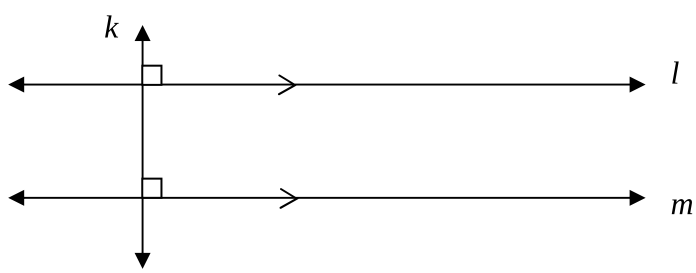
Since line *j* is perpendicular to line *l*, and line *k* is also perpendicular to line *l*, we can conclude that line *j* is parallel to line *k*. We use these arrows on the lines to show that $\overleftrightarrow{j} \parallel \overleftrightarrow{k}$

- **Proposition 13: Theorem 3.4**

  If a line is perpendicular to one of two parallel lines, it is perpendicular to the other also.



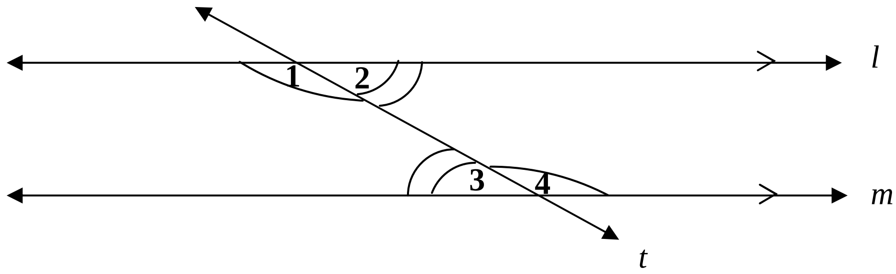
Given that lines *l* and *m* are parallel, and that line *k* is perpendicular to line *l*,



we know that line *k* is also perpendicular to line *m*.

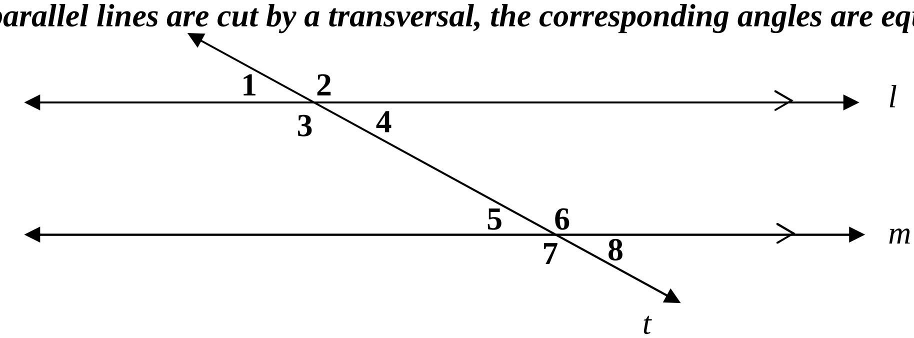
- **Proposition 14: Theorem 3.5**

  If two parallel lines are cut by a transversal, the alternate interior angles are equal.



Parallel lines *l* and *m* are being cut by the transversal line, *t*. From this theorem, we know $\angle 1 \cong \angle 4$, and $\angle 2 \cong \angle 3$.

C1: ***If two parallel lines are cut by a transversal, the corresponding angles are equal.***



The following angle pairs are congruent:

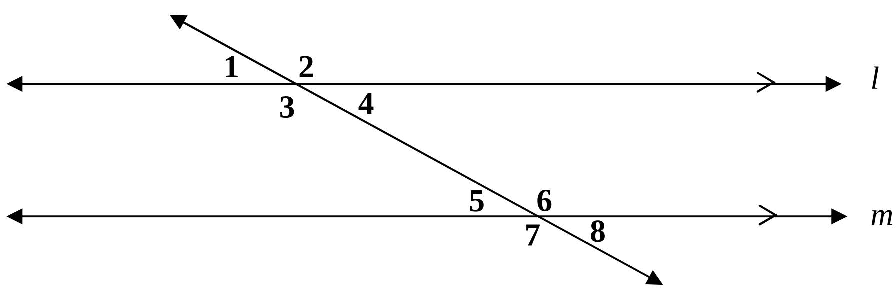
$$\angle 1 \cong \angle 5$$

$$\angle 3 \cong \angle 7$$

$$\angle 2 \cong \angle 6$$

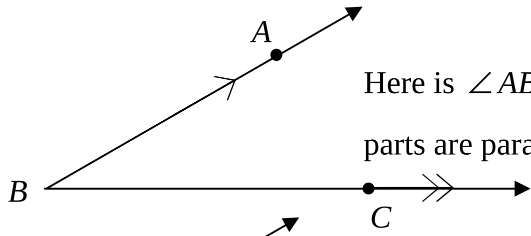
$$\angle 4 \cong \angle 8$$

C2: ***If two parallel lines are cut by a transversal, the interior angles on the same side of the transversal are supplementary.***

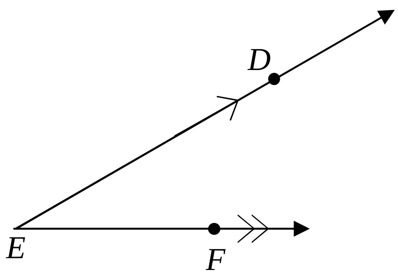


This means that $m\angle 3 + m\angle 5 = 180°$, and $m\angle 4 + m\angle 6 = 180°$ as well.

C3: ***If two angles have their sides respectively parallel, they are either equal or supplementary.***

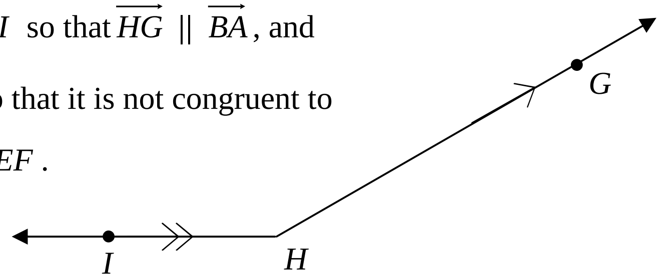


Here is $\angle ABC$. Let's make $\angle DEF$ so that its parts are parallel to each part of $\angle ABC$.

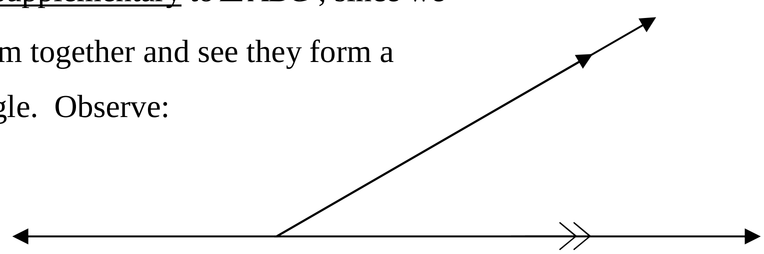


Notice that $\angle ABC \cong \angle DEF$.

Now let's make $\angle GHI$ so that $\overrightarrow{HG} \parallel \overrightarrow{BA}$, and $\overrightarrow{HI} \parallel \overrightarrow{BC}$, and also so that it is not congruent to either $\angle ABC$ or $\angle DEF$.

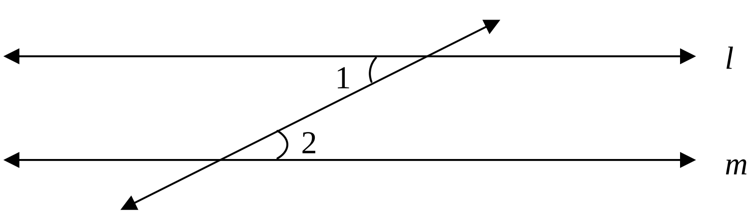


$\angle GHI$ is <u>supplementary</u> to $\angle ABC$, since we can put them together and see they form a straight angle. Observe:



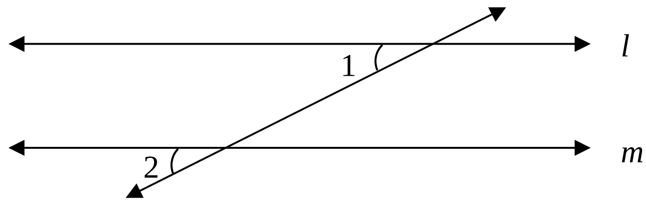
- **Proposition 15: Theorem 3.6**

  If two lines are cut by a transversal so that a pair of alternate interior angles is equal, the lines are parallel.



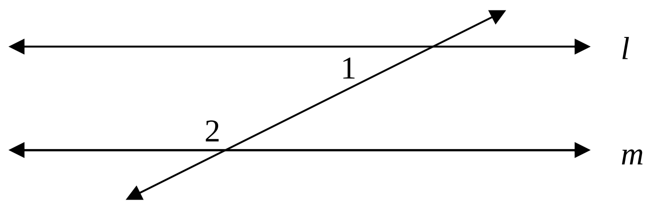
On the diagram at the left, because $\angle 1 \cong \angle 2$, we can conclude that lines *l* and *m* are parallel.

C1: ***If two lines are cut by a transversal so that a pair of corresponding angles is equal, the lines are parallel.***



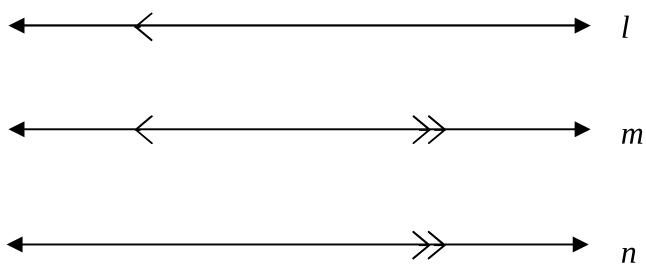
Because $\angle 1 \cong \angle 2$, we can conclude that lines *l* and *m* are parallel.

C2: ***If two lines are cut by a transversal so that the interior angles on the same side of the transversal are supplementary, the lines are parallel.***



If we show that the measures of angle 1 and angle 2 add to $180°$, then we know lines *l* and *m* are parallel.

C3: ***Two lines parallel to a third line are parallel to each other.***

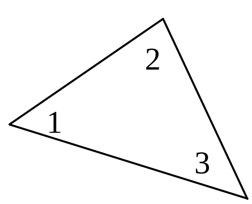


Since $\overleftrightarrow{l} \parallel \overleftrightarrow{m}$, and $\overleftrightarrow{n} \parallel \overleftrightarrow{m}$, we know that $\overleftrightarrow{l} \parallel \overleftrightarrow{n}$.

- **Proposition 16: Theorem 3.7**

  The sum of the angles of a triangle is equal to a straight angle or $180°$**.**

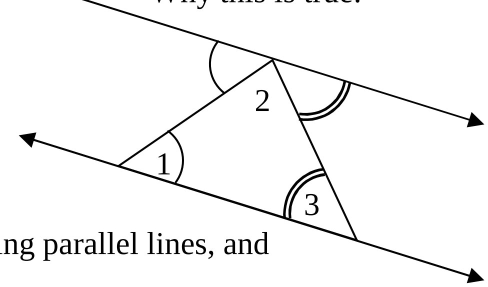
For any triangle,



All the angle measures will add to $180°$.

That is, $m\angle 1 + m\angle 2 + m\angle 3 = 180°$.

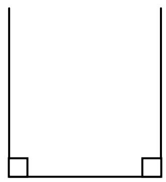
Why this is true:



Using parallel lines, and Proposition 14 on alternate interior angles, we see that $\angle 2$ and angles congruent to $\angle 1$, and $\angle 3$, form a straight angle, $180°$.

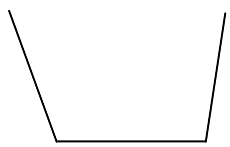
C1: ***A triangle can have but one right angle or one obtuse angle.***

Try to make a triangle with two right angles.



It will never be able to close. From this we know a triangle can have only one right angle.

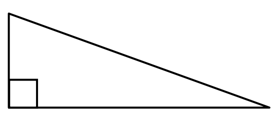
Similarly, if we try to make a triangle with two obtuse angles,

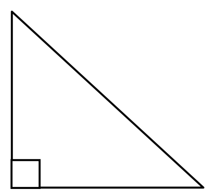


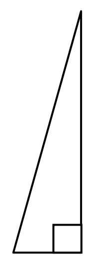
we see it is impossible to do; so a triangle can have at most one obtuse angle.

C2: ***The acute angles of a right triangle are complementary.***

In any right triangle,

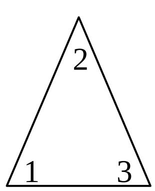


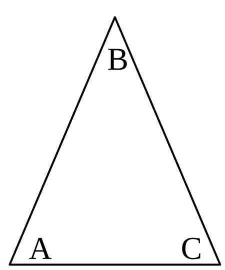




the total sum of the angles is $180°$. If we subtract the measure of the right angle from the total, we see that $180° - 90° = 90°$. The other two angles have to add to $90°$, which means that they are complementary.

C3: ***If two angles of one triangle are equal, respectively, to two angles of another triangle, the third angles are equal.***





Given:

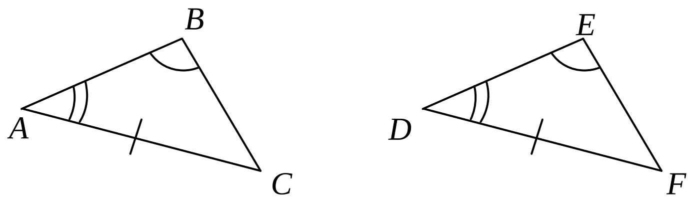
$\angle 1 \cong \angle A$

$\angle 2 \cong \angle B$

Conclusion:

$\angle 3 \cong \angle C$

C4: ***If two triangles have a side, adjacent angle, and the opposite angle of the one equal, respectively, to the corresponding parts of the other, the triangles are congruent.***



Since:

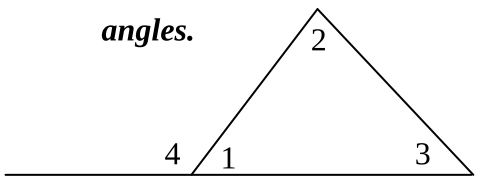
$\angle B \cong \angle E$

$\angle A \cong \angle D$

$\overline{AC} \cong \overline{DF}$

Conclude: $\Delta ABC \cong \Delta DEF$

C5: ***An exterior angle of a triangle is equal to the sum of the two opposite interior angles.***



Notice that $m\angle 1 + (m\angle 2 + m\angle 3) = 180°$, and

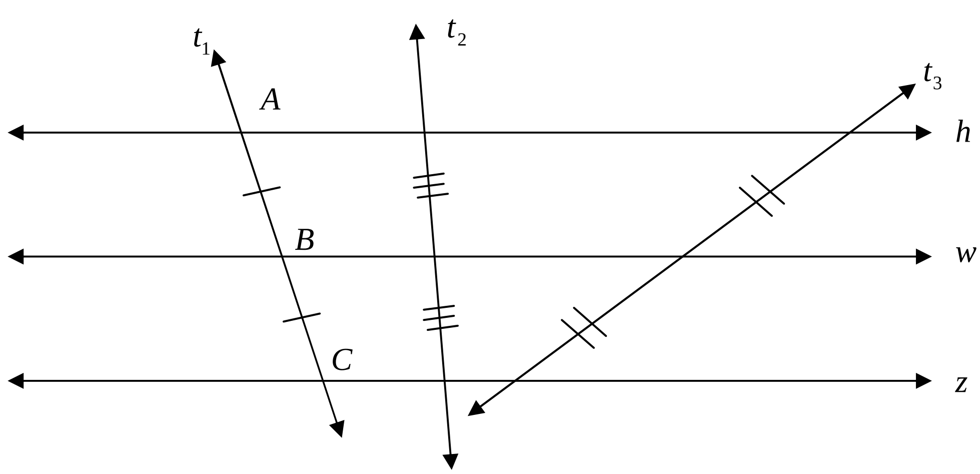
$m\angle 1 + (m\angle 4) = 180°$

Since angle 1 is supplementary to both things in parentheses, we know that $m\angle 4 = m\angle 2 + m\angle 3$.

- **Proposition 17: Theorem 3.8**

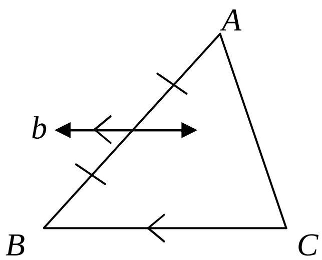
If three or more parallels intercept equal segments on one transversal, they intercept equal segments on every transversal.

We'll show this with just three parallel lines. Given $\overleftrightarrow{h} \parallel \overleftrightarrow{w} \parallel \overleftrightarrow{z}$, cut by transversal line t_1 at points *A*, *B*, and *C*, such that $\overline{AB} \cong \overline{BC}$, then any transversal will have congruent segments between lines *h*, *w*, and *z*, as shown with transversal lines t_2 and t_3.

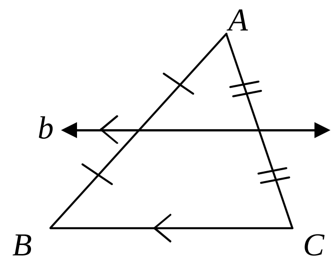


C1: ***If a line bisects one side of a triangle and is parallel to a second side, it bisects the third side also.***

(Figure 1)

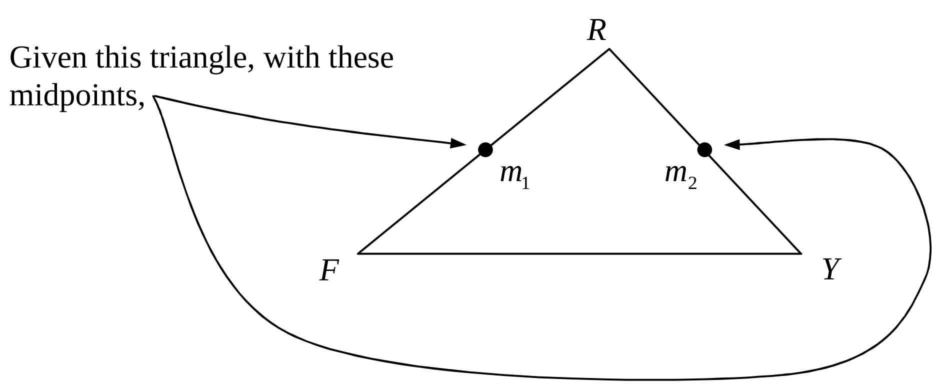


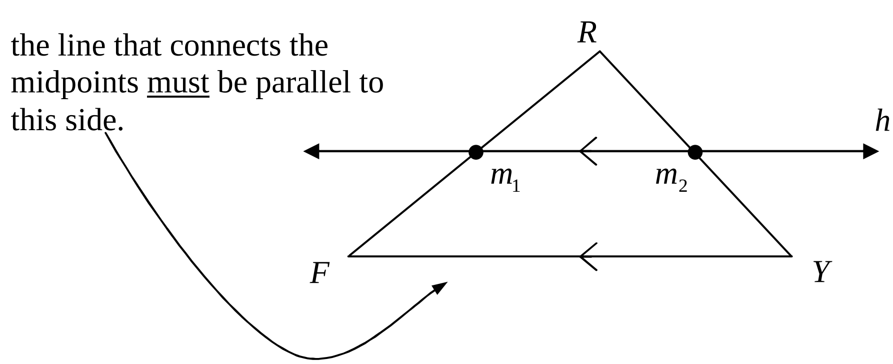
(Figure 2)



In figure 1, we see that line b is parallel to $\overline{BC}$ and line b also bisects line segment $\overline{AB}$. Because of this, we conclude the following (in figure 2): line $\overleftrightarrow{b}$ also bisects $\overline{AC}$.

C2: ***If a line connects the midpoints of two sides of a triangle, it is parallel to the third.***

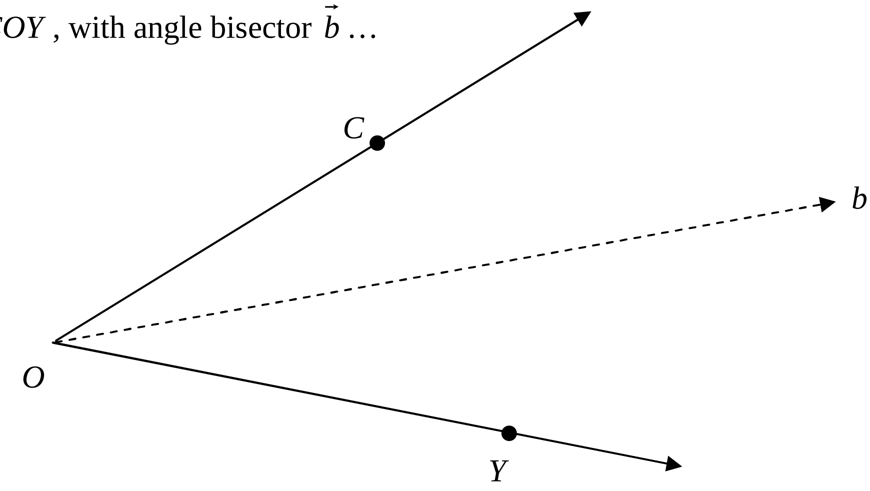




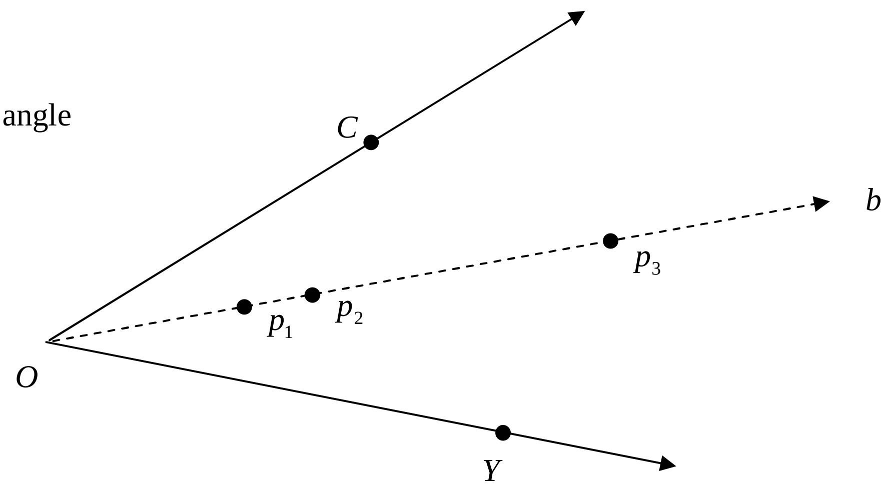
Proposition 18: Theorem 3.9

Any point on the bisector of an angle is equidistant from the sides of the angle.

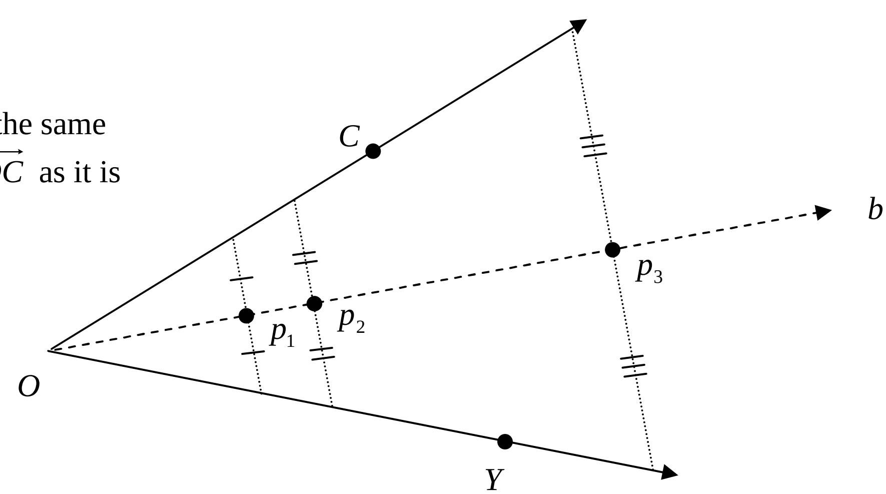
Given any angle, say $\angle COY$, with angle bisector $\vec{b}$ …



…and points on angle bisector $\vec{b}$ …

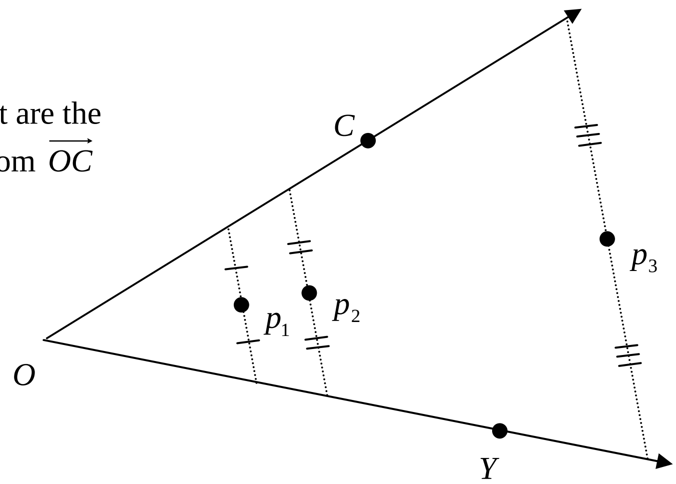


…each point is the same distance from $\overrightarrow{OC}$ as it is from $\overrightarrow{OY}$.

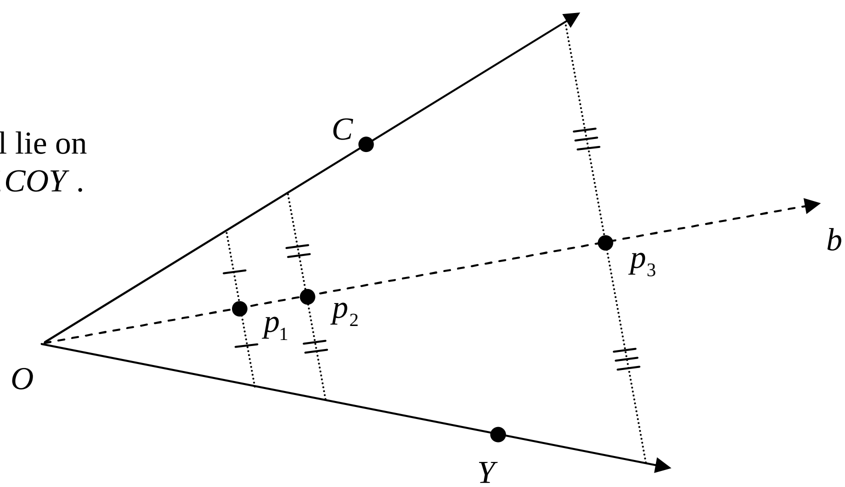


Converse: Any point equidistant from the sides of an angle is on the bisector of the angle.

Given points that are the same distance from $\overrightarrow{OC}$ as from $\overrightarrow{OY}$ …

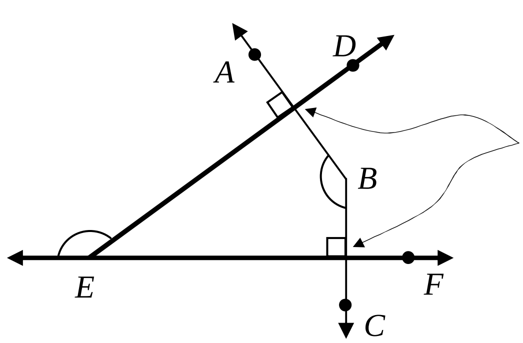


…those points all lie on the bisector of $\angle COY$.

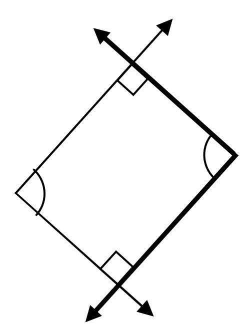


- **Proposition 19: Theorem 3.10**

  If two angles have their sides, respectively, perpendicular, they are either equal or supplementary.

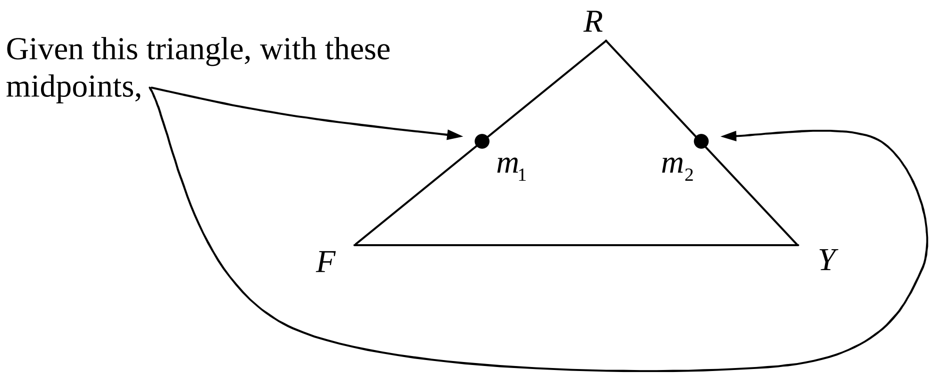


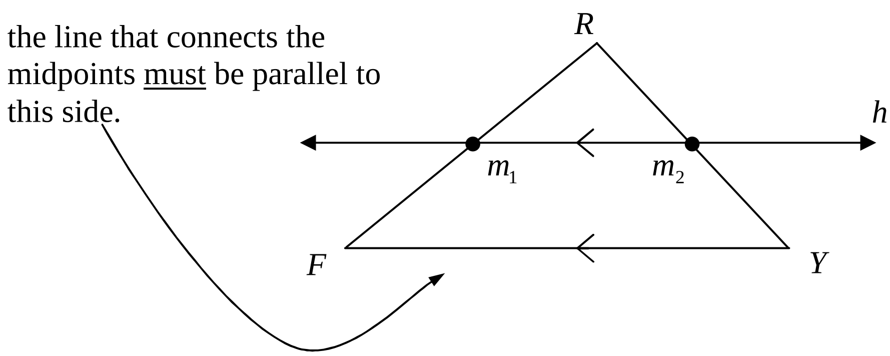
On the left, $\angle ABC$ and $\angle DEF$ have perpendicular sides, and the two angles are supplementary. On the right, these angles have perpendicular sides and equal measure.



- **Proposition 20: Theorem 3.11**

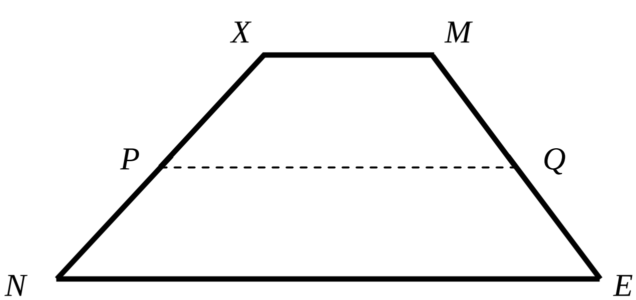
  A line segment connecting the midpoints of two sides of a triangle is parallel to the third side and equal to half of it.





Also, the length of line segment, $\overline{m_1m_2}$ is half the length of line segment $\overline{FY}$.

C: ***The median of a trapezoid is parallel to the bases and equal to half their sum***.



In trapezoid *XMEN*, the median, line segment $\overline{PQ}$ is such that:

$\overline{XM}$, $\overline{PQ}$, $\overline{NE}$ are all parallel,

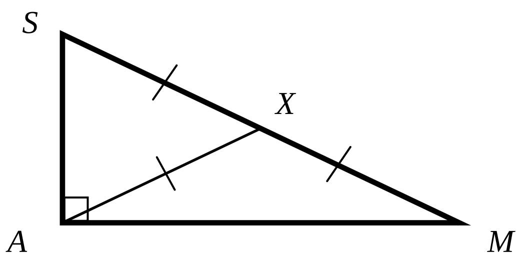
and *PQ* is half of (*XM* + *NE*).

Note: To really see this in action, visit the following website:
http://www.mathopenref.com/trapezoidmedian.html

- **Proposition 21: Theorem 3.12**

In a right triangle, the median to the hypotenuse is equal to half the hypotenuse.

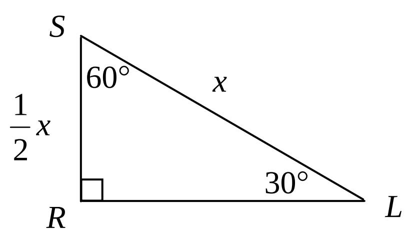
Given a right triangle ΔSAM with hypotenuse $\overline{SM}$ whose median is $\overline{AX}$ …



…AX is half of SM.

C1: ***In a*** $30^0 - 60^0$ ***right triangle, the side opposite the*** $30°$ ***angle is half the hypotenuse.***

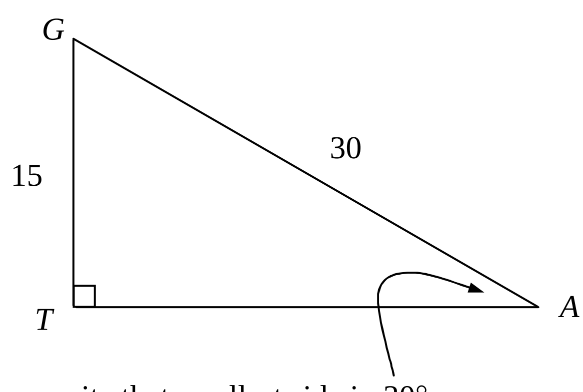
Here is ΔSLR with angle measures $30°, 60°, 90°$.



This corollary states that the side opposite the $30°$ angle (side $\overline{SR}$) is half the length of the hypotenuse (side $\overline{SL}$). This can be proven by drawing the median to the hypotenuse, and showing the resulting acute triangle is also equilateral.

C2: ***If the hypotenuse of a right triangle is double one of the sides, then the acute angle opposite that side is 30 degrees.***

Given a right triangle whose hypotenuse is twice the length of one of the sides…

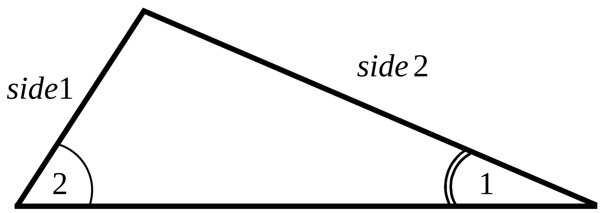


…the angle opposite that smallest side is $30°$.

This is the converse of the last corollary.

- **Proposition 22: Theorem 3.13**

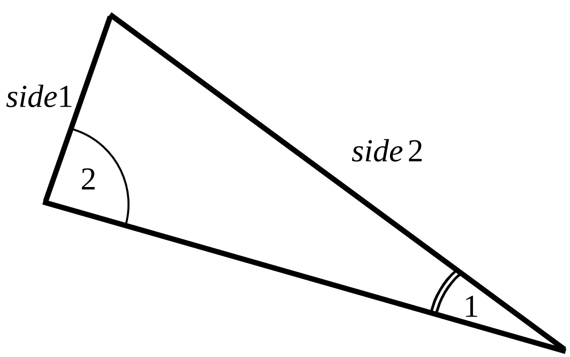
  If two sides of a triangle are unequal, the angles opposite these sides are unequal and the angle opposite the larger side is the larger angle.



Given side 1 opposite angle 1, and side 2 opposite angle 2, where side 1 and side 2 are different lengths. Notice also that side 2 is larger than side 1. By the proposition, we conclude that angle 1 and angle 2 are also different measures, and that angle 2 is larger than angle 1.

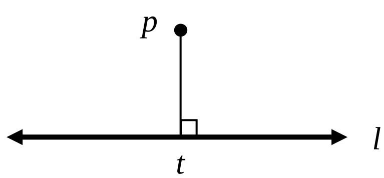
- **Proposition 23: Theorem 3.14**

  If two angles of a triangle are unequal, the sides opposite these angles are unequal and the side opposite the larger angle is the larger side.



Given side 1 opposite angle 1, and side 2 opposite angle 2, where angle 1 and angle 2 are different measures. Notice that angle 2 is larger than angle 1. By the proposition, we conclude that side 1 and side 2 are also different lengths, and that side 2 is larger than side 1.

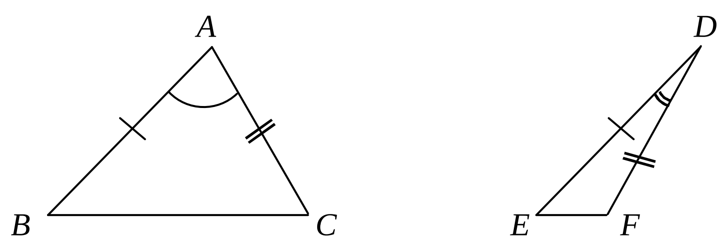
C: ***The shortest segment that can be drawn from a given point to a given line is the perpendicular from the point to the line.***



Line segment $\overline{pt}$ is perpendicular to line *l*, and $\overline{pt}$ is the shortest line segment that can be drawn from point *p* that touches line *l*.

Proposition 24: Theorem 3.15

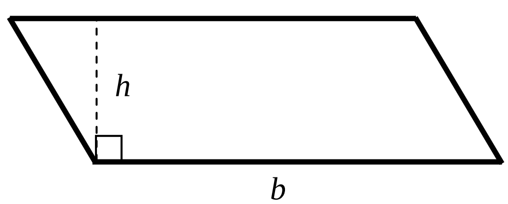
If two sides of a triangle are equal, respectively, to two sides of another triangle, but the included angle of the first is greater than the included angle of the second, then the third side of the first is greater than the third side of the second.



Triangles ΔABC and ΔDEF have two sets of congruent sides. However, since $\angle A$ is larger than $\angle D$, this tells us that side $\overline{BC}$ is larger than side $\overline{EF}$.

- **Proposition 25: Theorem 4.1**

The area of a parallelogram is equal to the product of its base and its altitude.



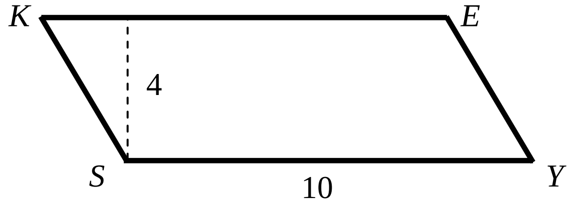
(Altitude means height.)

Area = base × height

$$A = b \times h$$

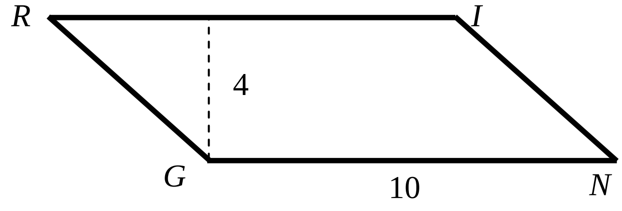
C1: ***Parallelograms with equal bases and equal altitudes are equal in area.***

Parallelogram *KEYS*



Area of *KEYS* = $b \times h = 10 \times 4 = 40\ units^2$

Parallelogram *RING*

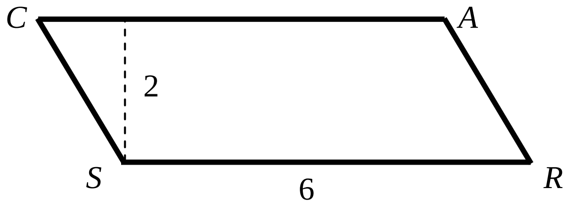


Area of *RING* = $b \times h = 10 \times 4 = 40\ units^2$

C2: ***The areas of two parallelograms with equal altitudes have the same ratio as the ratio of their bases.***

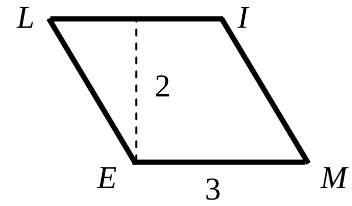
Consider two parallelograms whose heights are equal…

Parallelogram *CARS*



Area of $CARS = 6 \times 2 = 12\ units^2$

Parallelogram *LIME*



Area of $LIME = 3 \times 2 = 6\ units^2$

…Now take the ratio of the areas, and compare it to the ratio of the bases. They will be the same.

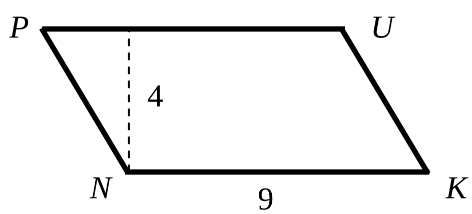
$$\frac{\text{Area of } CARS}{\text{Area of } LIME} = \frac{12\ units^2}{6\ units^2} = \textcircled{2}$$

$$\frac{\text{Base of } CARS}{\text{Base of } LIME} = \frac{6}{3} = \textcircled{2}$$

C3: ***The areas of two parallelograms with equal bases have the same ratio as the ratio of their altitudes.***

This time, start with two parallelograms whose bases are equal, and find their areas.

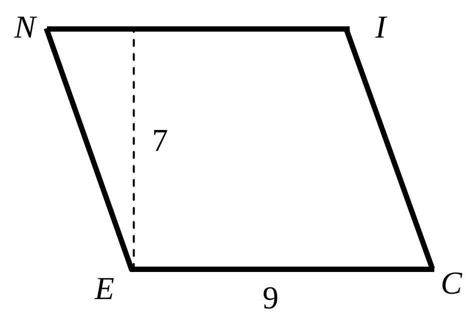
Parallelogram *PUNK*



Area of $PUNK = 9 \times 4 = 36\ units^2$

Area of $NICE = 9 \times 7 = 63\ units^2$

Parallelogram *NICE*



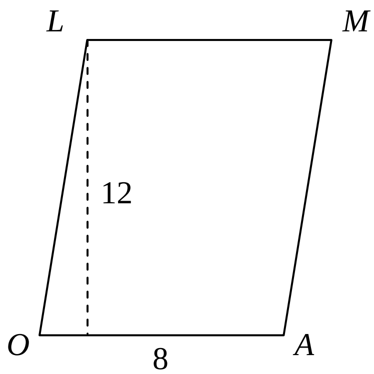
Take the ratio of the areas, and compare it with the ratio of the heights. They will always be equal.

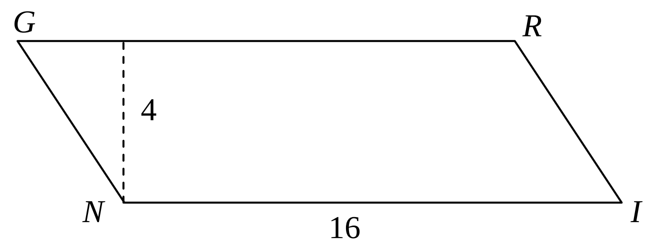
$$\frac{\text{Area of } PUNK}{\text{Area of } NICE} = \frac{36\ units^2}{63\ units^2} = \left(\frac{4}{7}\right)$$

$$\frac{\text{Height of } PUNK}{\text{Height of } NICE} = \left(\frac{4}{7}\right)$$

C4: ***The areas of two parallelograms have the same ratio as the product of their bases and altitudes.***

Given any two parallelograms…





$A = 16 \times 4 = 64\ units^2$

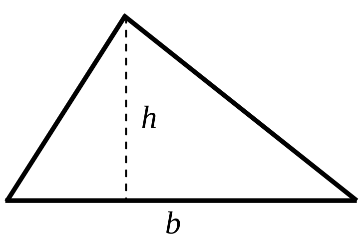
$A = 8 \times 12 = 96\ units^2$

…the following ratios are equal:

$$\frac{96\ units^2}{64\ units^2} = \frac{8\ units \times 12\ units}{16\ units \times 4\ units}$$

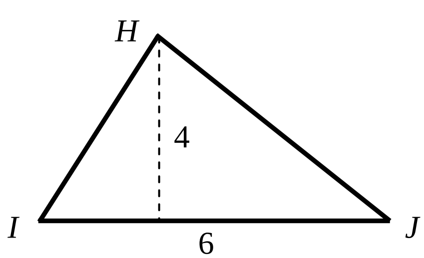
- **Proposition 26: Theorem 4.2**

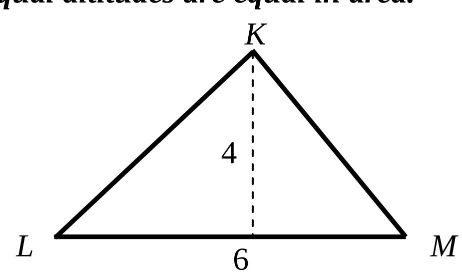
The area of a triangle is equal to half the product of its base and its altitude.



$Area = \frac{1}{2}(b \times h)$

C1: ***Two triangles with equal bases and equal altitudes are equal in area.***

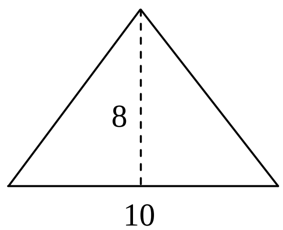


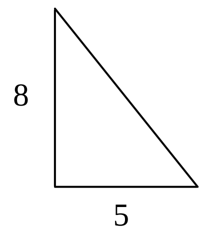


Area of ΔHIJ = Area of ΔKLM = $\frac{1}{2}(b \times h) = \frac{1}{2}(6 \times 4) = 12\,units^2$

C2: ***The areas of triangles with equal altitudes have the same ratio as their bases.***

Remember, altitude means height. Take two triangles with equal heights, and find their area.





$$A = \frac{1}{2}(b \times h) \qquad\qquad A = \frac{1}{2}(b \times h)$$

$$= \frac{1}{2}(10 \times 8) \qquad\qquad = \frac{1}{2}(5 \times 8)$$

$$= 40\, units^2 \qquad\qquad = 20\, units^2$$

This corollary says that the <u>ratio</u> of the areas…

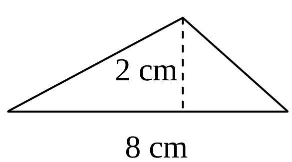
$$\left(\frac{40\, units^2}{20\, units^2} = 2\right)$$

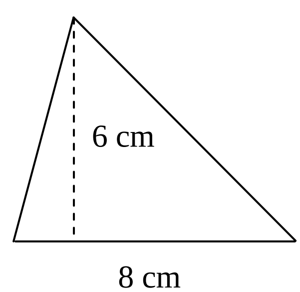
…is always the same as the ratio of the bases.

$$\left(\frac{10}{5} = 2\right)$$

C3: ***The areas of triangles with equal bases have the same ratio as their altitudes.***

This is similar to C2 above, except now we start with triangles whose bases are equal, and find their area.





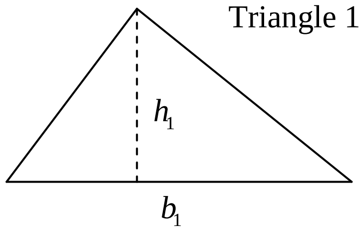
$$A = \frac{1}{2}(b \times h) = \frac{1}{2}(8cm)(2cm) = 8\,cm^2 \qquad A = \frac{1}{2}(b \times h) = \frac{1}{2}(8\,cm)(6cm) = 24\,cm^2$$

$$\text{Ratio of areas} = \frac{8\text{ cm}^2}{24\text{ cm}^2} = \frac{1}{3} \qquad \text{Ratio of heights} = \frac{2\text{ cm}}{6\text{ cm}} = \frac{1}{3}$$

These ratios are always equal.

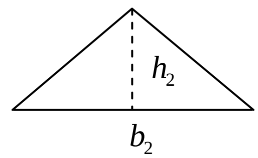
C4: ***The areas of two triangles have the same ratio as the product of their bases and altitudes.***

Triangle 1



Area of Triangle 1 = $A_1 = \frac{1}{2} \times b_1 \times h_1$

Triangle 2



Area of Triangle 2 = $A_2 = \frac{1}{2} \times b_2 \times h_2$

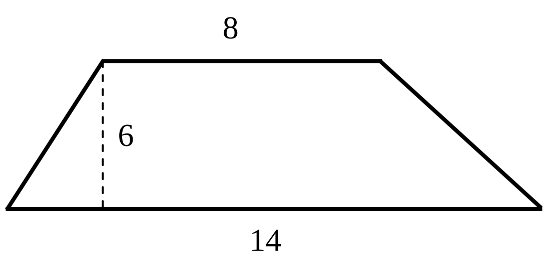
This corollary allows us to conclude,

$$\frac{A_1}{A_2} = \frac{\cancel{1/2} \times b_1 \times h_1}{\cancel{1/2} \times b_2 \times h_2} = \frac{b_1 \times h_1}{b_2 \times h_2}$$

- **Proposition 27: Theorem 4.3**

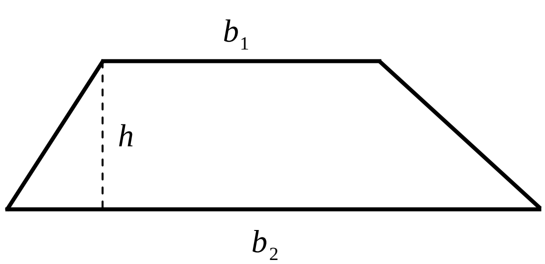
The area of a trapezoid is equal to half the product of the sum of its bases and its altitude.

Start with any trapezoid, say one with the following measurements,



The area is,

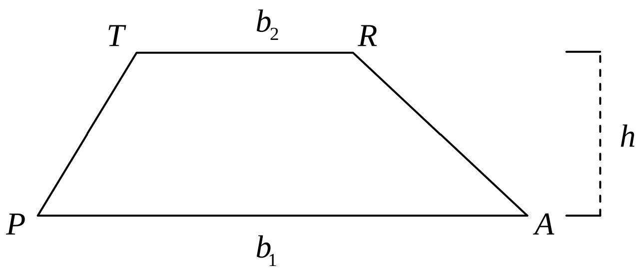
$$A = \frac{1}{2}((8+14)\times 6)$$
$$= \frac{1}{2}(22\times 6) = \frac{1}{2}(132) = 66\,units^2$$



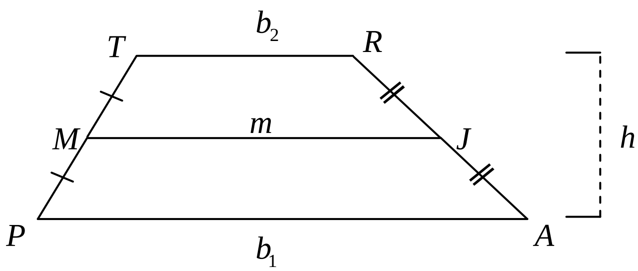
In general, the area of a trapezoid is,

$$A = \frac{1}{2}[(b_1 + b_2)\times h]$$

C: ***The area of a trapezoid is equal to the product of its altitude and the segment connecting the midpoints of the legs.***



Begin with a trapezoid, in this case trapezoid *TRAP*, whose altitude, or height, is *h*. Notice that the legs of *TRAP* are segments $\overline{TP}$ and $\overline{RA}$.

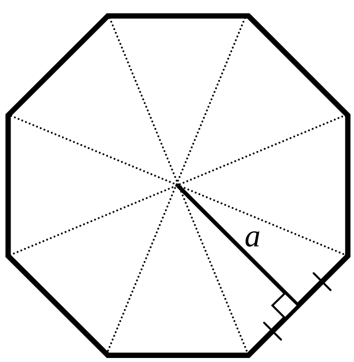


The midpoints of the legs are labeled points *M* and *J*. Now, the area of trapezoid *TRAP* is given by the length of the segment, $\overline{MJ}$, times the height, *h*, or $(A = m \times h)$. This is because

$$m = \frac{(b_1 + b_2)}{2}$$

Proposition 28: Theorem 4.4

The area of a regular polygon is equal to half the product of its apothem and its perimeter.



Perimeter = *P*

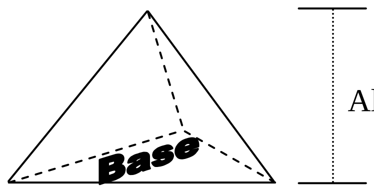
Remember, a <u>regular polygon</u> is one whose sides are all the same length. All regular polygons are made of equally sized triangles. The <u>apothem</u> is the segment that has one endpoint in the center of the polygon, and the other endpoint bisecting one of the sides to form right angles. This regular octagon has apothem *a*, and perimeter *P*. By this proposition, its area is given by,

$$A = \frac{1}{2} \times a \times P$$

Note: The formula is found by adding the area of each triangle that makes up the polygon.

- **Proposition 29: Theorem 4.5**

  The volume of a triangular pyramid is equal to one-third the product of its altitude and the area of its base.



$$A = \frac{1}{3} \times (Area\, of\ Base) \times a$$

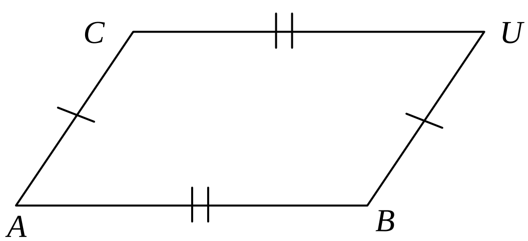
Note: the altitude is the height of the pyramid.

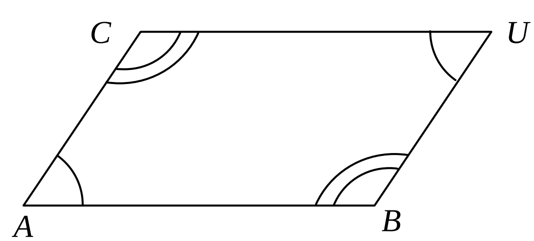
- **Proposition 30: Theorem 5.1**

  The opposite sides of a parallelogram are equal and the opposite angles are equal.

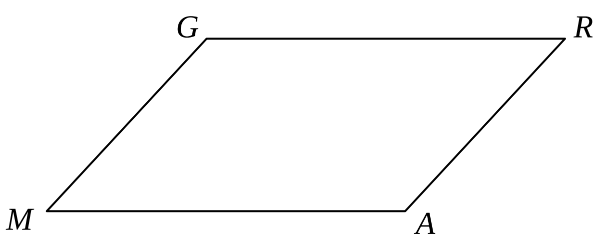
In parallelogram *CUBA*,

opposite sides are equal, and opposite angles are equal.

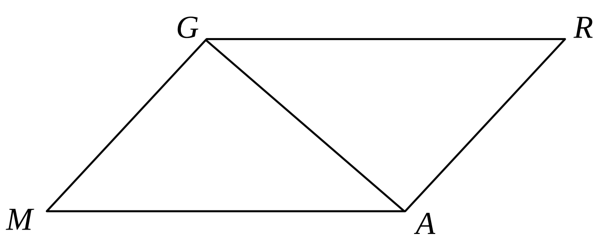




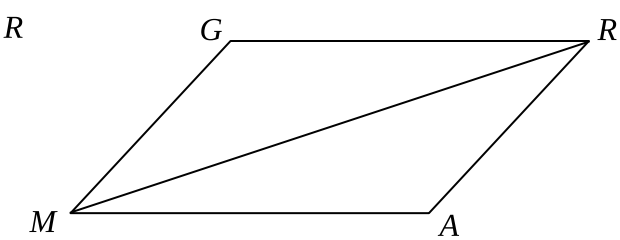
C1: ***A parallelogram is divided into two congruent triangles by either diagonal.***



Given parallelogram *GRAM*…

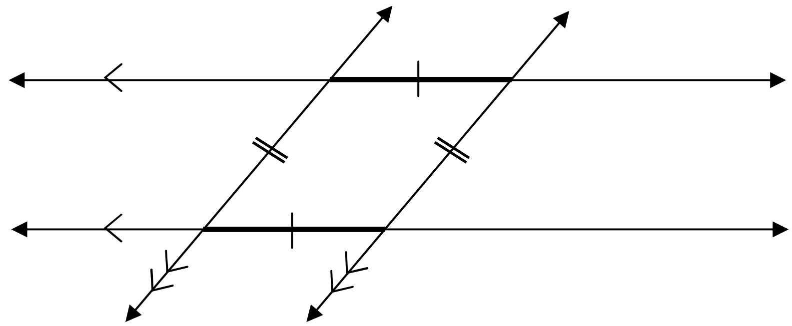


…diagonal $\overline{GA}$ is such that $\Delta MGA \cong \Delta RAG$.



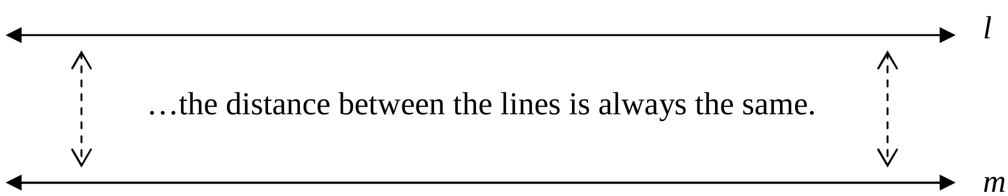
Likewise, diagonal $\overline{MR}$ forms $\Delta MGR \cong \Delta RAM$.

C2: ***Segments of parallel lines cut off by a second set of parallel lines are equal.***



C3: ***Two parallel lines are everywhere equidistant.***

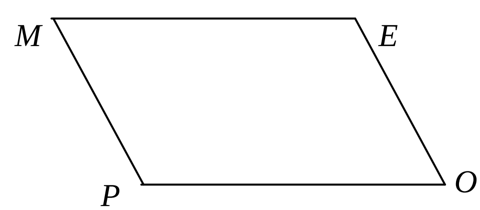
Given that lines l and m are parallel…



- **Proposition 31: Theorem 5.2**

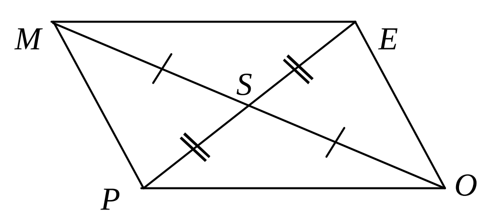
The diagonals of a parallelogram bisect each other.

Begin with a parallelogram, *MEOP*.



Draw its diagonals, $\overline{PE}$ and $\overline{MO}$.

The diagonals will always cross at each other's midpoint.

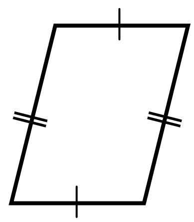


Note: You can prove this by showing that $\Delta MSP \cong \Delta OSE$ (this is not the only way).

- **Proportion 32: Theorem 5.3**

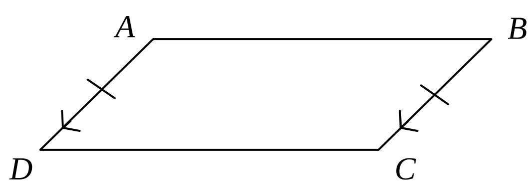
  If both sets of opposite sides of a quadrilateral are equal (congruent), the quadrilateral is a parallelogram.

  This is a parallelogram, since the sides across from each other are the same.



- **Proposition 33: Theorem 5.4**

  If one pair of opposite sides of a quadrilateral is both equal and parallel, the quadrilateral is a parallelogram.



Notice in quadrilateral *ABCD*:

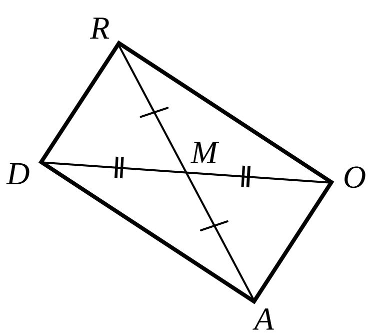
$\overline{AD}$ and $\overline{BC}$ are opposite sides.

$\overline{AD} \cong \overline{BC}$, and $\overline{AD} \parallel \overline{BC}$.

Therefore, *ABCD* is a parallelogram.

- **Proposition 34: Theorem 5.5**

  If the diagonals of a quadrilateral bisect each other, the quadrilateral is a parallelogram.



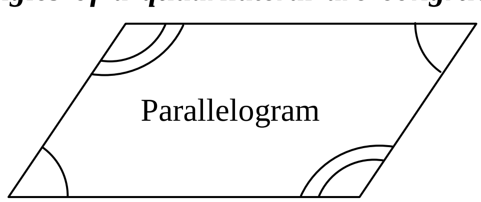
Here we have quadrilateral *ROAD*, with diagonals $\overline{RA}$ and $\overline{DO}$ that cross at the point *M*.

Since $\overline{RM} \cong \overline{AM}$ and $\overline{DM} \cong \overline{OM}$, the diagonals bisect each other.

Because of this, *ROAD* is a parallelogram.

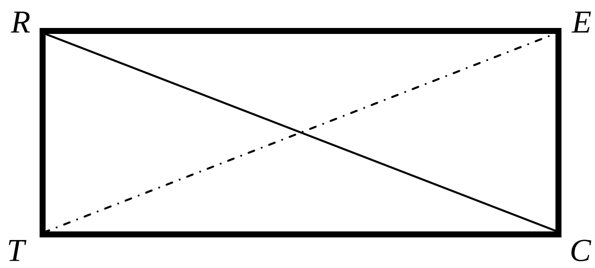
- **Proposition 35: Theorem 5.6**

  If both sets of opposite angles of a quadrilateral are congruent, the quadrilateral is a parallelogram.



- **Proposition 36: Theorem 5.7**

  If the diagonals of a parallelogram are congruent, then the parallelogram is a rectangle.

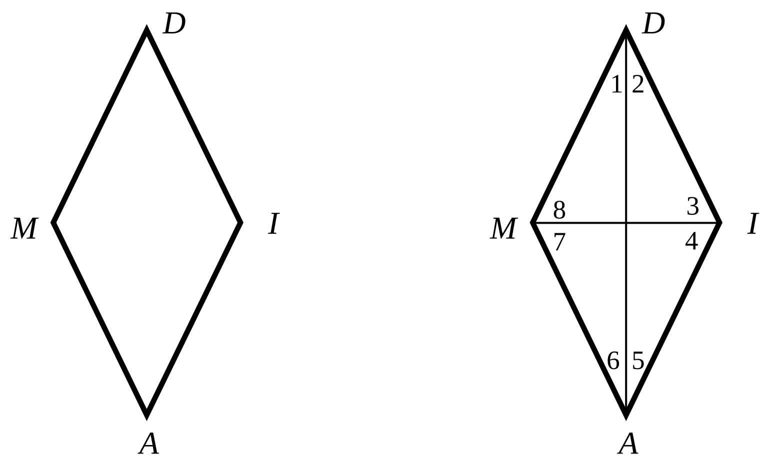


We know that *RECT* is a rectangle if we can show that $\overline{RC} \cong \overline{ET}$.

- **Proposition 37: Theorem 5.8**

  If the diagonals of a parallelogram bisect the angles of the parallelogram, then the parallelogram is a rhombus.

  Parallelogram *DIAM* has diagonals $\overline{DA}$ and $\overline{MI}$ that form angles 1 through 8.

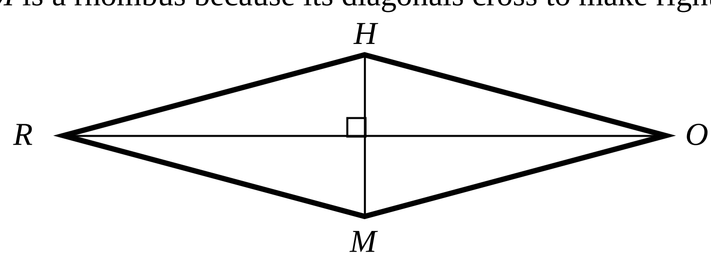


If :
$\angle 1 \cong \angle 2$,
$\angle 3 \cong \angle 4$,
$\angle 5 \cong \angle 6$, and
$\angle 7 \cong \angle 8$,
then *DIAM* is a rhombus.

- **Proposition 38: Theorem 5.9**

  If the diagonals of a parallelogram are perpendicular, then the parallelogram is a rhombus.

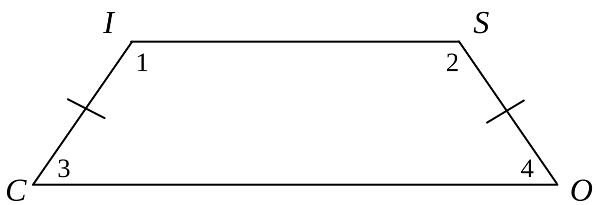
  Parallelogram *RHOM* is a rhombus because its diagonals cross to make right angles.

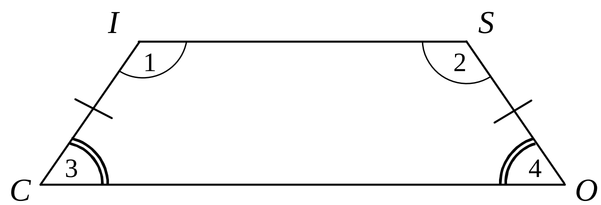


- **Proposition 39: Theorem 5.10**

  Base angles of an isosceles trapezoid are congruent.

  Given isosceles trapezoid, *ISOC*, with angles 1, 2, 3, and 4, we know…

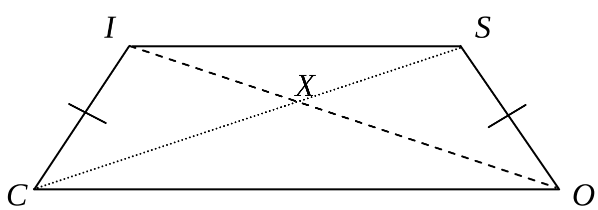




$\angle 1 \cong \angle 2$, and $\angle 3 \cong \angle 4$

C: ***The diagonals of an isosceles trapezoid are congruent.***

Isosceles trapezoid *ISOC* has diagonals $\overline{IO}$ and $\overline{SC}$.

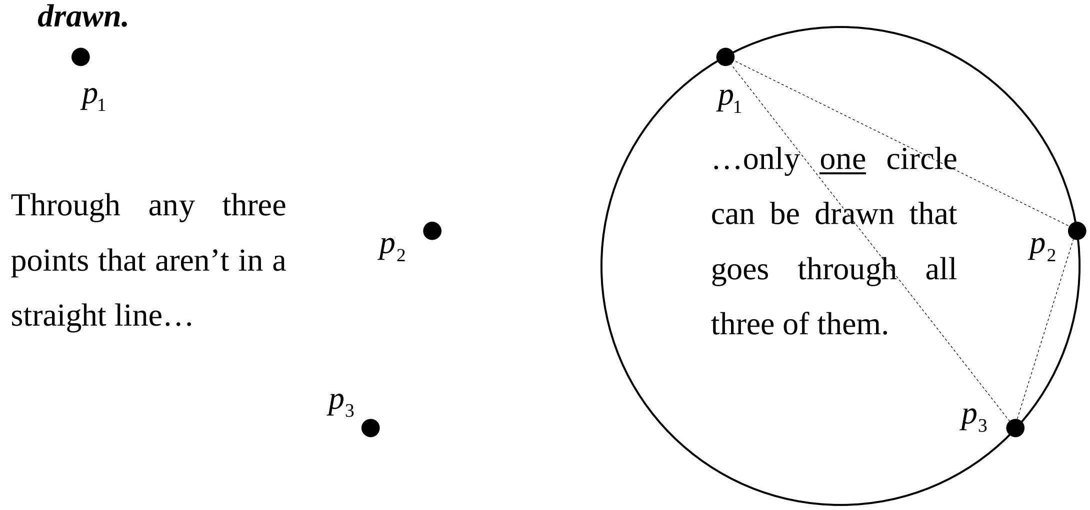


Because this is an <u>isosceles</u> trapezoid, we know that $\overline{IO} \cong \overline{SC}$.

Note: This is proven by showing $\Delta ICO \cong \Delta SOC$

- **Proposition 40: Theorem 6.1**

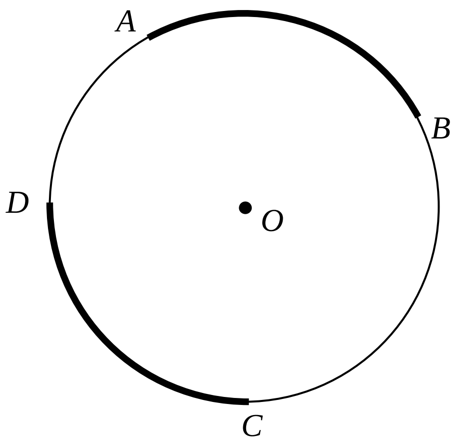
  Through any three given points not in a straight line, one and only one circle can be drawn.



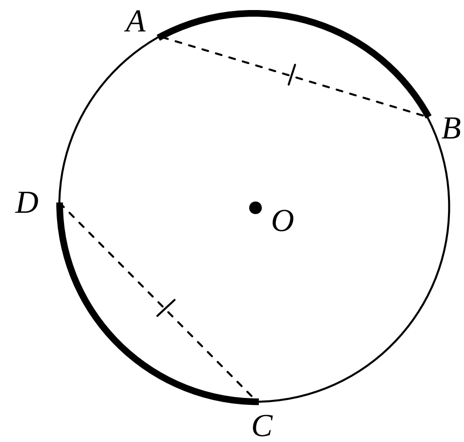
Note: This circle is the <u>circumcircle</u> to the triangle the points form.

- **Proposition 41: Theorem 6.2**

  In the same circle, or in equal circles, if two arcs are equal, their chords are equal.



Given circle O with congruent arcs $\overset{\frown}{AB} \cong \overset{\frown}{CD}$ …



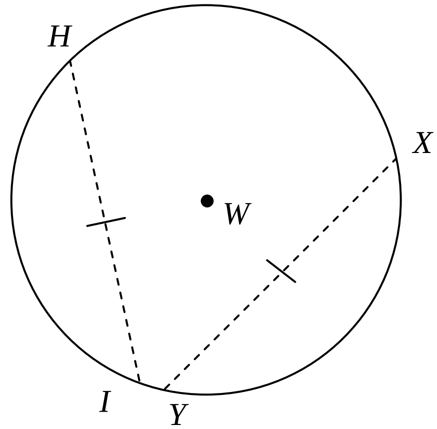
…conclude $\overline{AB} \cong \overline{CD}$

We can prove this by drawing in the radii $\overline{OA}$, $\overline{OB}$, $\overline{OC}$, and $\overline{OD}$. Then we can show that $\Delta ABO \cong \Delta CDO$.

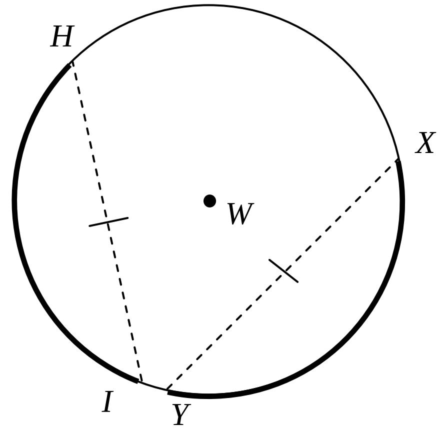
- **Proposition 42: Theorem 6.3**

  In the same circle, or in equal circles, if two chords are equal, their arcs are equal.

Given circle W with chords $\overline{HI}$ and $\overline{XY}$ …



…if $\overline{HI} \cong \overline{XY}$ …

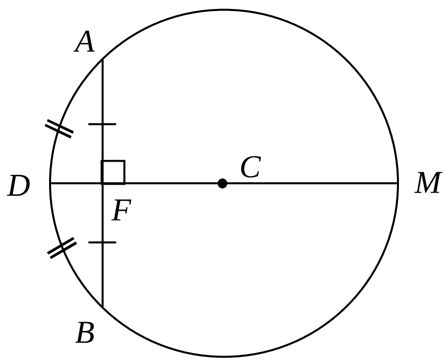


…conclude $\overset{\frown}{HI} \cong \overset{\frown}{XY}$.

We can prove this in a similar way to proposition 41.

- **Proposition 43: Theorem 6.4**

  A diameter perpendicular to a chord bisects the chord and its arcs.

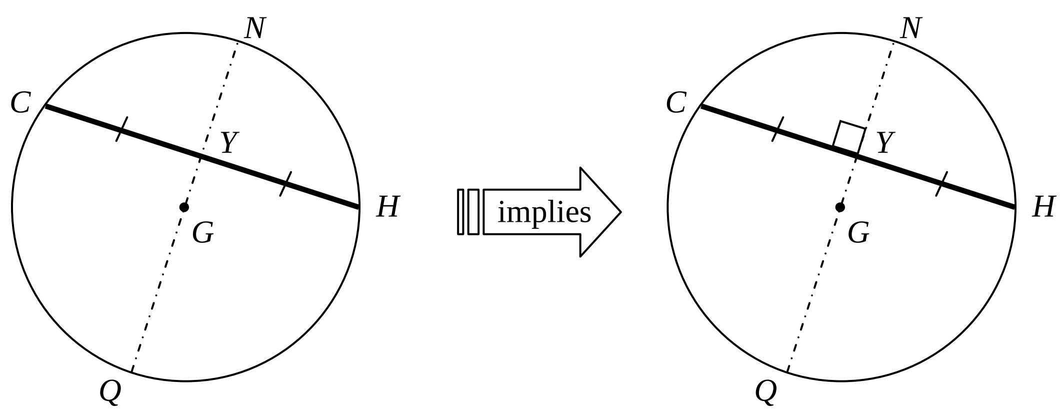


Given circle C that has diameter $\overline{DM}$, perpendicular to chord $\overline{AB}$ at point F, we know that the chord is split into two equal sections: $\overline{AF} \cong \overline{BF}$. We also know arc $\widehat{ADB}$ is cut into two congruent arcs as well: arc $\widehat{AD}$, and arc $\widehat{BD}$.

- **Proposition 44: Theorem 6.5**

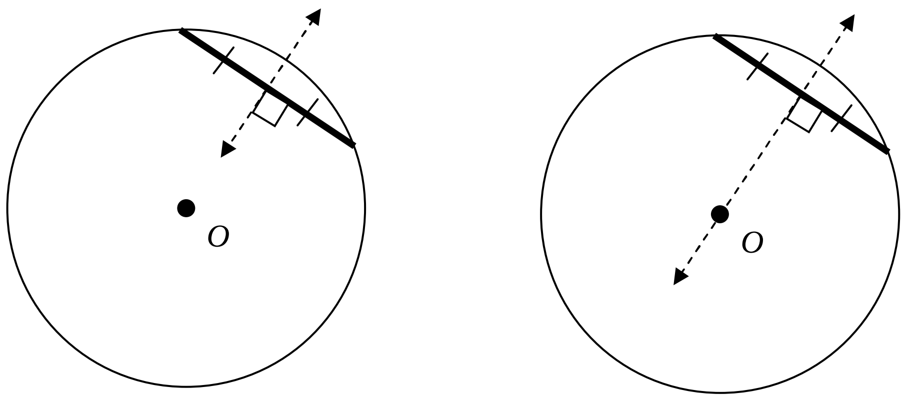
  A diameter that bisects a chord (not a diameter) is perpendicular to the chord.

In circle G, since its diameter, $\overline{NQ}$, bisects chord $\overline{CH}$, we know that the diameter and the chord are perpendicular $\left(\overline{NQ} \perp \overline{CH}\right)$.



$$\overline{NQ} \perp \overline{CH}$$

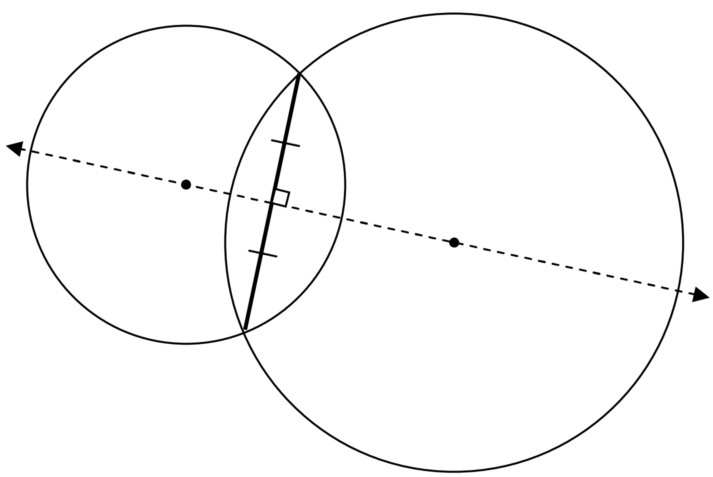
C: ***The perpendicular bisector of a chord passes through the center of the circle.***

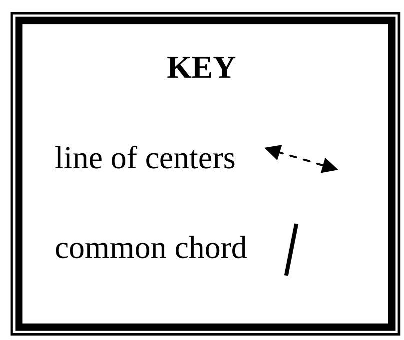


- **Proposition 45: Theorem 6.6**

  If two circles intersect, the line of centers is the perpendicular bisector of their common chord.

intersecting circles

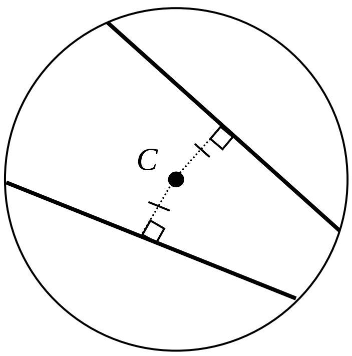


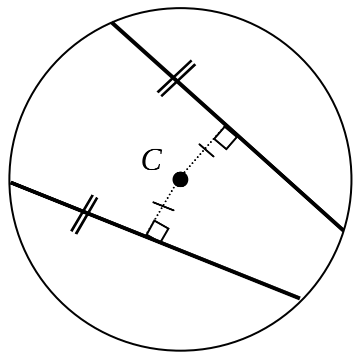


- **Proposition 46: Theorem 6.7**

  In the same circle, or in equal circles, chords equidistant from the center are equal.

  Given that two chords are each the same distance from the center of a circle, you can conclude that the chords are congruent.



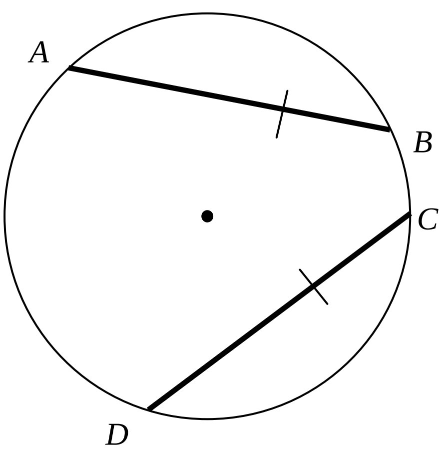


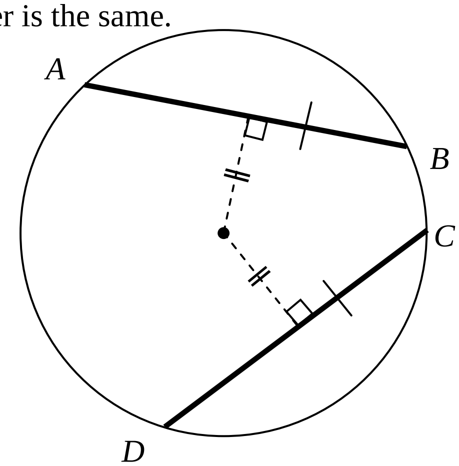
Note: The proof of this is shown in Unit 1, Lesson 7, using congruent triangles, and by the fact that the dotted lines above are perpendicular bisectors, and cut the chords in half.

- **Proposition 47: Theorem 6.8**

  In the same circle, or in equal circles, equal chords are equidistant from the center.

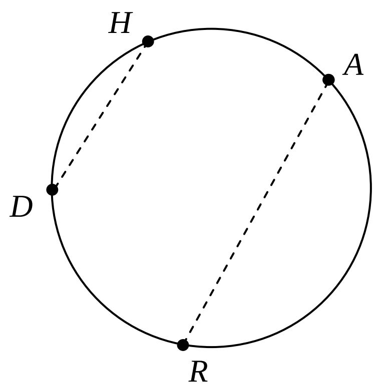
Given that $\overline{AB} \cong \overline{CD}$, you can conclude that the distance of each chord from the center is the same.





- **Proposition 48: Theorem 6.9**

  In the same circle, or in equal circles, if two minor arcs are unequal, the larger arc has the larger chord.



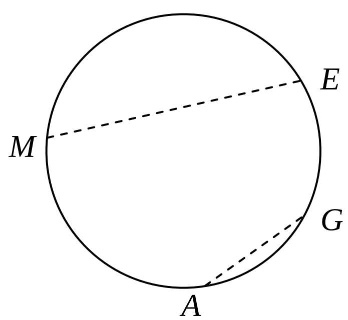
Notice the arcs first.

Arc $\overset{\frown}{AR}$ is larger than arc $\overset{\frown}{HD}$.

Because of this, we see that chords $AR > HD$.

- **Proposition 49: Theorem 6.10**

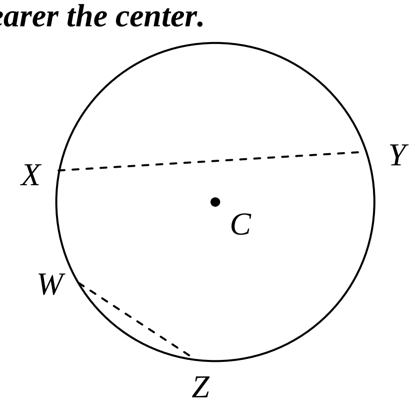
  In the same circle, or in equal circles, if two chords are unequal, the larger chord has the larger minor arc.



Notice that in the circle, chords $\overline{ME}$ and $\overline{GA}$ are not equal, and that $ME > GA$. From this theorem, we then conclude that the arc measures are unequal in the same way. That is, $m\overset{\frown}{ME} > m\overset{\frown}{GA}$

- **Proposition 50: Theorem 6.11**

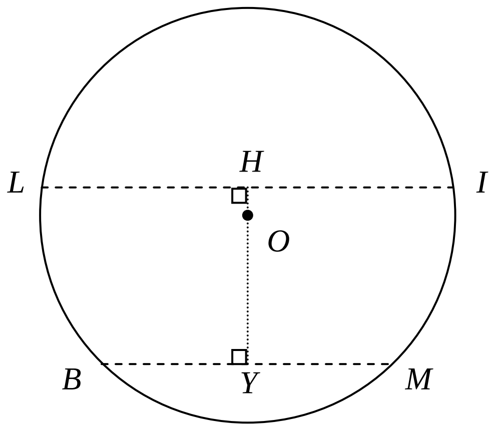
  In the same circle, or in equal circles, if two chords are unequal, the larger chord is nearer the center.



As you can see, $\overline{XY}$ is closer to C than $\overline{WZ}$ is.

- **Proposition 51: Theorem 6.12**

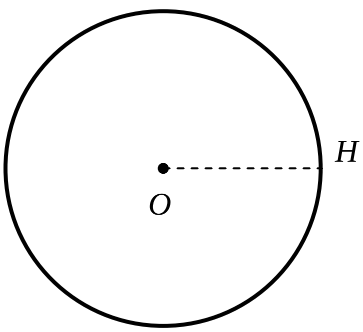
  In the same circle, or in equal circles, if two chords are unequally distant from the center, the one nearer the center is larger.



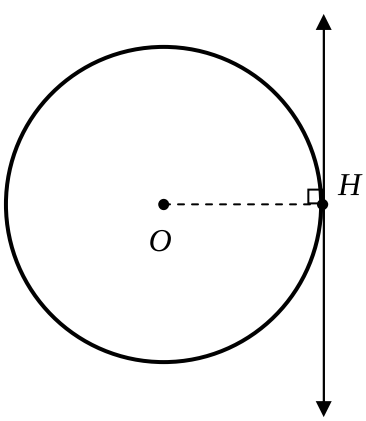
The distances between chords $\overline{LI}$ and $\overline{BM}$, and the center of circle *O*, shown by line segments $\overline{OH}$ and $\overline{OY}$, are not equal. In fact, because $\overline{LI}$ is nearer to the center than $\overline{BM}$ is, we can conclude that $LI > BM$.

- **Proposition 52: Theorem 6.13**

  A line perpendicular to a radius at its outer extremity is tangent to the circle.



Given circle *O* with radius $\overline{OH}$ …

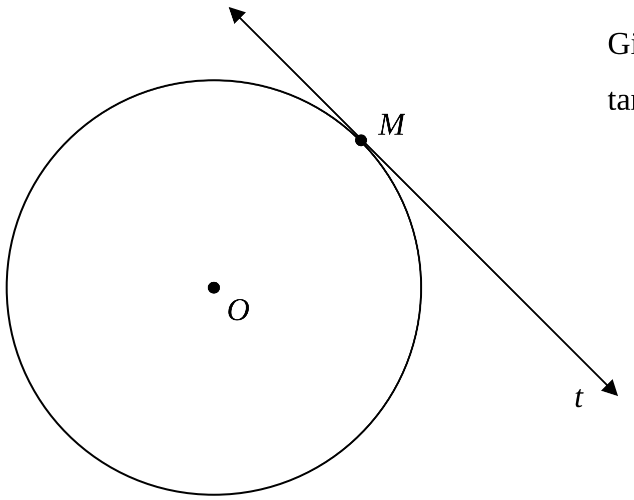


…if we draw a perpendicular line that crosses $\overline{OH}$ at *H*, its outer extremity, that line will be tangent* to circle *O*.

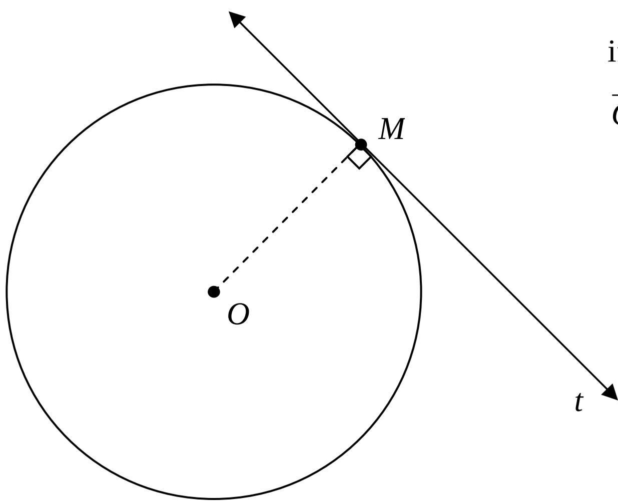
* tangent means that the line touches the circle exactly once, and does not go through it.

- **Proposition 53: Theorem 6.14**

The tangent to a circle at a given point is perpendicular to the radius drawn to that point.



Given circle *O* and line *t* where line *t* is tangent to circle *O* at the point *M*,

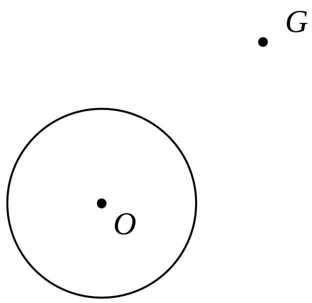


if we draw radius $\overline{OM}$, then line *t* and $\overline{OM}$ must be perpendicular.

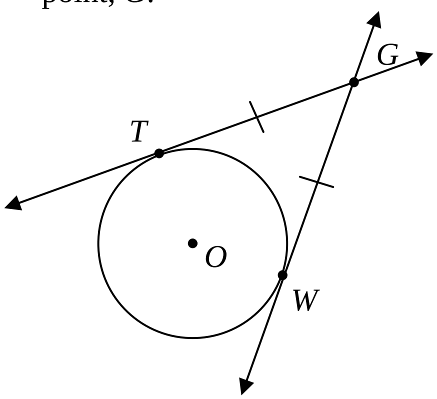
- **Proposition 54: Theorem 6.15**

  Two tangents to a circle from an outside point are equal and make equal angles with the line joining that point to the circle.

Start with a circle, *O*, and a point, *G*, outside the circle.

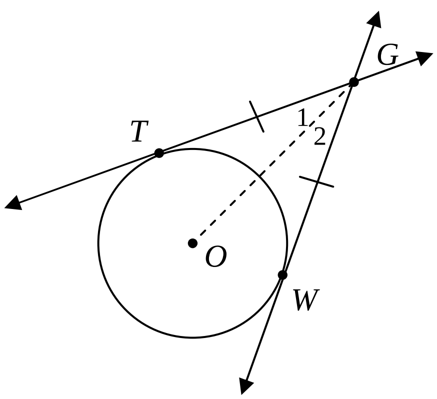


Next, observe the two lines that are tangent to the circle, and that pass through the point, *G*.



We conclude that $\overline{TG} \cong \overline{WG}$.

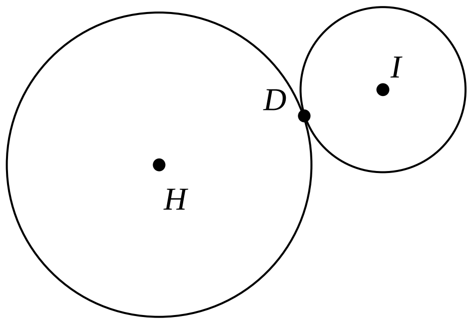
Furthermore, if we draw the line segment $\overline{GO}$, which forms angles 1 and 2,



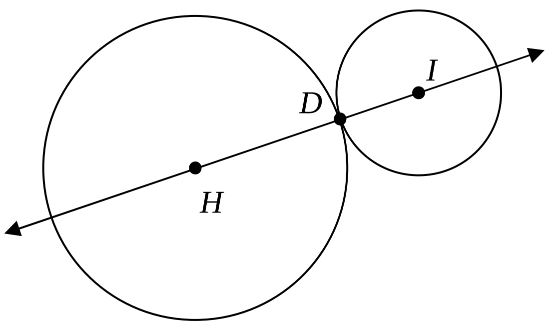
we will have $\angle 1 \cong \angle 2$.

Proposition 55: Theorem 6.16

If two circles are tangent externally, the line of centers passes through the point of contact.



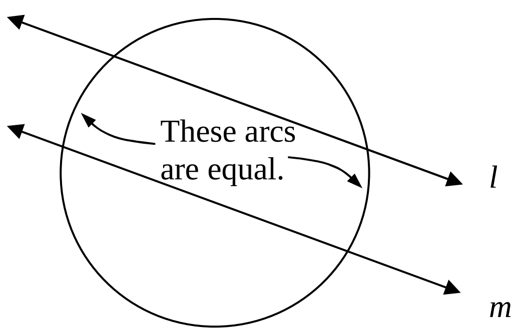
These circles are tangent externally, which is a fancy way of saying they touch without crossing. The point of contact is point *D*, where they touch. This proposition says that line $\overleftrightarrow{HI}$, the line of centers, will also pass through point *D*, as shown below.



- **Proposition 56: Theorem 6.17**

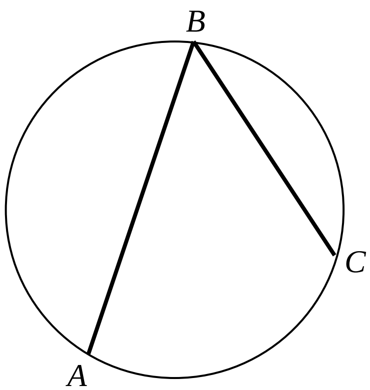
Two parallel lines intercept equal arcs on a circle.

Given line *l* is parallel to line *m*, we know that



- **Proposition 57: Theorem 7.1**

  An inscribed angle has the same measure as half of its intercepted arc.

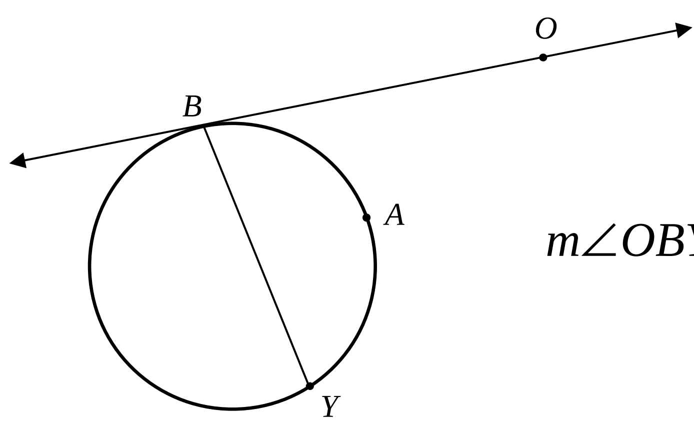


(*Note*: An <u>inscribed angle</u> is an angle whose vertex lies on the circle, and whose legs also intersect the circle.)

$$m\angle ABC = \frac{1}{2} \times m\overset{\frown}{AC}$$

- **Proposition 58: Theorem 7.2**

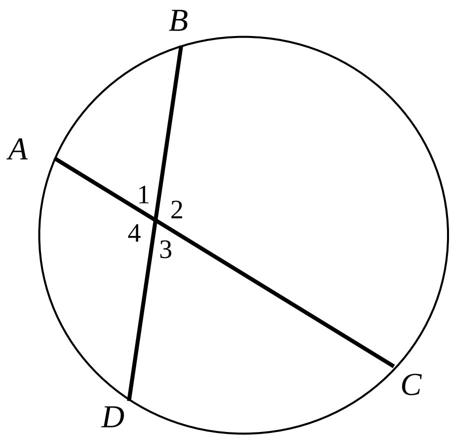
  An angle formed by a tangent and a chord from the point of contact has the same measure as half its intercepted arc.



$$m\angle OBY = \frac{1}{2} \times m\overset{\frown}{BAY}$$

- **Proposition 59: Theorem 7.3**

An angle formed by two intersecting chords has the same measure as half the sum of the intercepted arcs.



$$m\angle 1 = \frac{1}{2}\left(m\widehat{AB} + m\widehat{DC}\right)$$

$$m\angle 2 = \frac{1}{2}\left(m\widehat{AD} + m\widehat{BC}\right)$$

$$m\angle 3 = \frac{1}{2}\left(m\widehat{AB} + m\widehat{DC}\right)$$

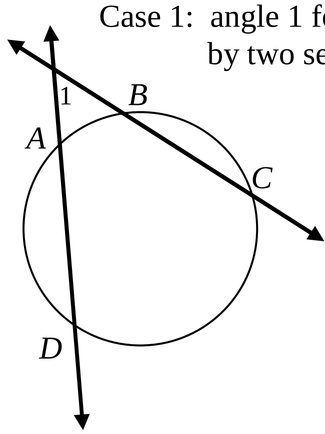
$$m\angle 4 = \frac{1}{2}\left(m\widehat{AD} + m\widehat{BC}\right)$$

Notice that angles 1 and 3 are congruent, and angles 2 and 4 are congruent. This should make sense, since these angle pairs are each vertical angles.

- **Proposition 60: Theorem 7.4**

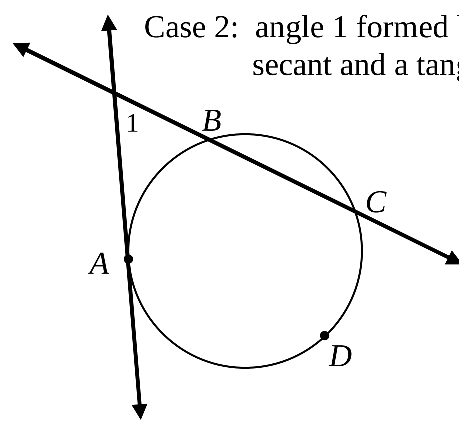
An angle formed by two secants, by a secant and a tangent, or by two tangents intersecting outside the circle has the same measure as half the difference between the intercepted arcs. (*Note*: A secant is a line which intersects a circle at two points.)

Case 1: angle 1 formed by two secants



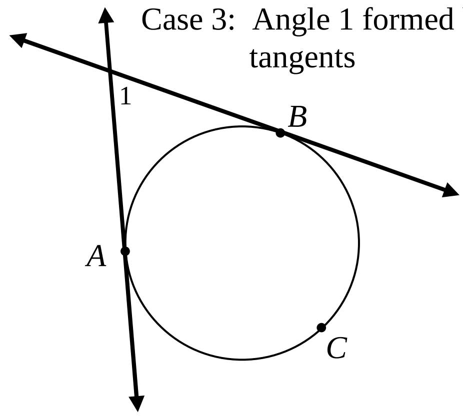
$$m\angle 1 = \frac{1}{2}\left(m\widehat{DC} - m\widehat{AB}\right)$$

Case 2: angle 1 formed by a secant and a tangent



$$m\angle 1 = \frac{1}{2}\left(m\widehat{ADC} - m\widehat{AB}\right)$$

Case 3: Angle 1 formed by two tangents

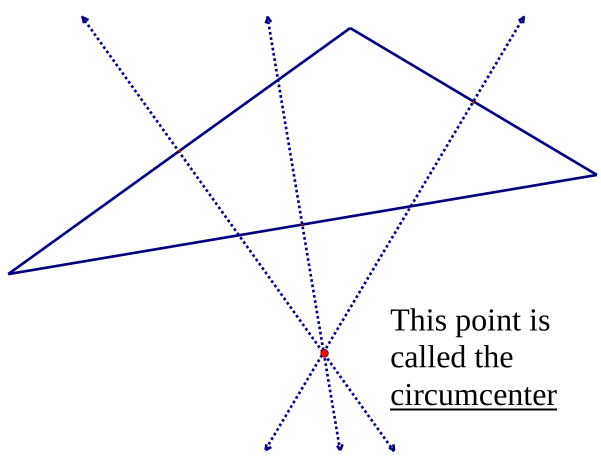


$$m\angle 1 = \frac{1}{2}\left(m\widehat{BCA} - m\widehat{AB}\right)$$

- **Proposition 61: Theorem 7.5**

  The perpendicular bisectors of the sides of a triangle are concurrent.

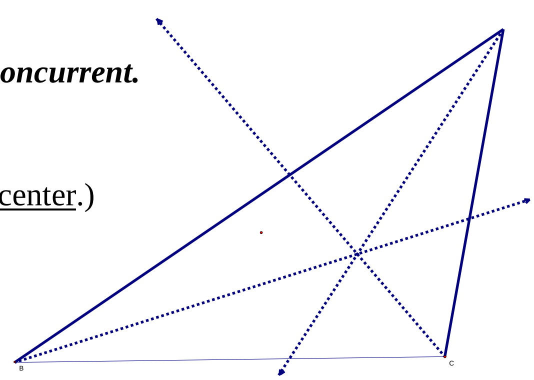
  (*Note*: <u>Concurrent</u> means that they all intersect at the same point.)



- **Proposition 62: Theorem 7.6**

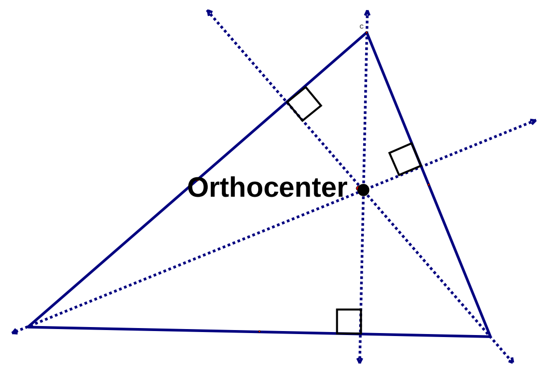
  The bisectors of the angles of a triangle are concurrent.

  (*Note*: the point of intersection is called the <u>incenter</u>.)



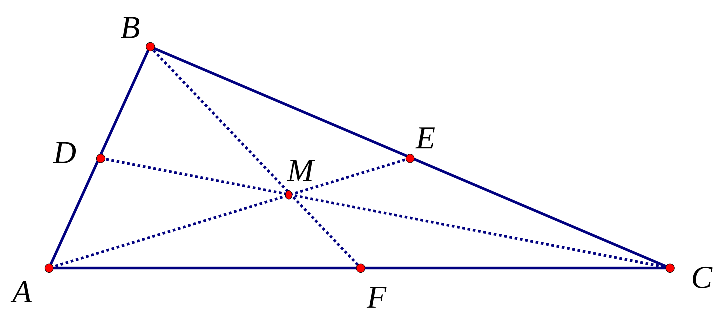
- **Proposition 63: Theorem 7.7**

  The altitudes of a triangle are concurrent.



Proposition 64: Theorem 7.8

The medians of a triangle intersect at a point that is two-thirds the distance from any vertex to the midpoint of the opposite side.



$$BM = \frac{2}{3} \times BF$$

$$CM = \frac{2}{3} \times CD$$

$$AM = \frac{2}{3} \times AE$$

Note: point *M* is called the centroid.

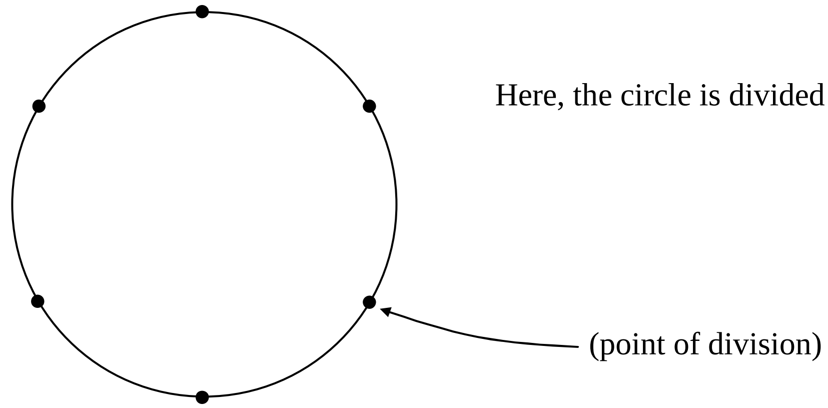
- **Proposition 65: Theorem 7.9**

  If a circle is divided into equal arcs, the chords of these arcs form a regular inscribed polygon.

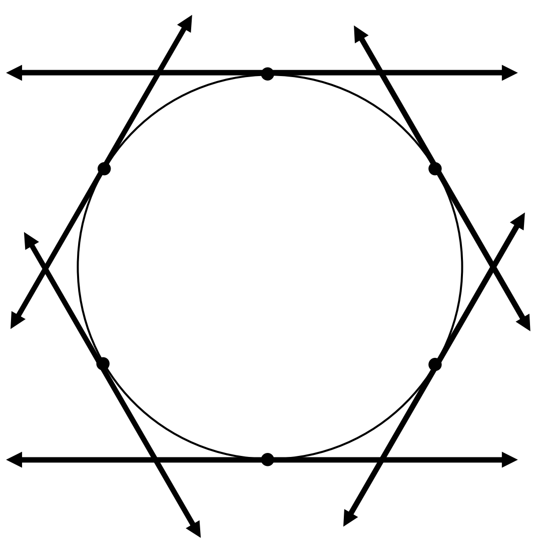
Here, we've divided a circle into 5 equal arcs. The chords connect to form a regular pentagon.

- **Proposition 66: Theorem 7.10**

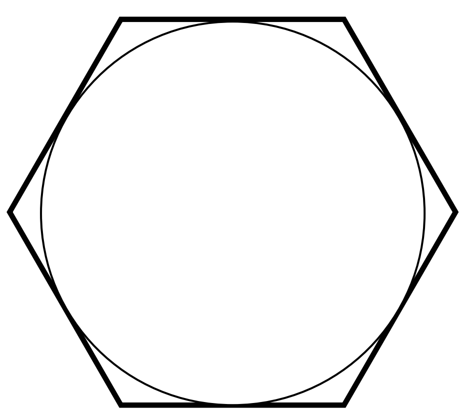
  If a circle is divided into equal arcs, the tangents at the point of division form a regular circumscribed polygon.



Here, the circle is divided into six equal arcs.



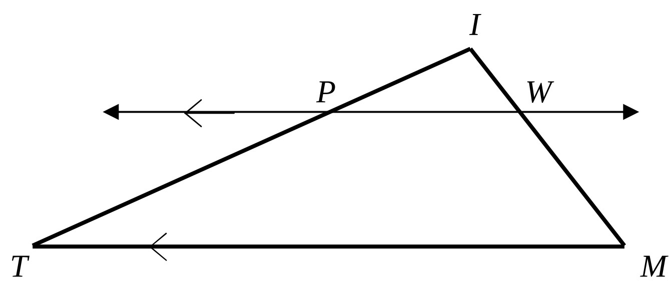
When we take the tangent lines to each division point…



…we see that the tangent lines cross to form a regular polygon – in this case, a regular hexagon.

- **Proposition 67: Theorem 7.11**

A line parallel to one side of a triangle and intersecting the other two sides, divides those sides proportionally.

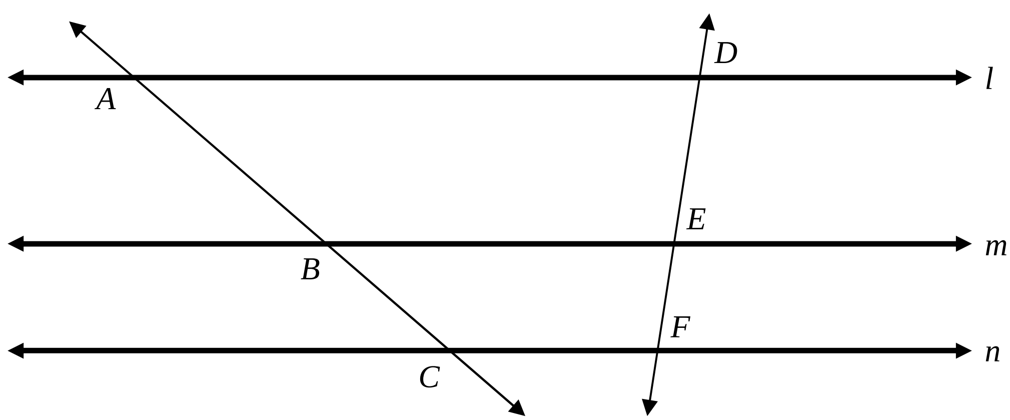


Here, ΔTIM is being intersected by line $\overleftrightarrow{PW}$, where $\overleftrightarrow{PW}$ is parallel to $\overleftrightarrow{TM}$. Because of this, the following is true:

$$\frac{IP}{PT} = \frac{IW}{WM}$$

- **Proposition 68: Theorem 7.12**

The segments cut off on two transversals by a series of parallels are proportional.

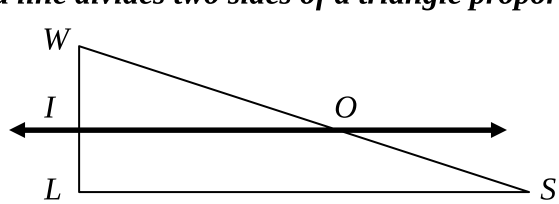


Given that lines *l*, *m*, *n* are all parallel, and are being cut by transversal lines $\overleftrightarrow{AC}$ and $\overleftrightarrow{DF}$, we know that the following ratios are equal:

$$\frac{AB}{BC} = \frac{DE}{EF}$$

- **Proposition 69: Theorem 7.13**

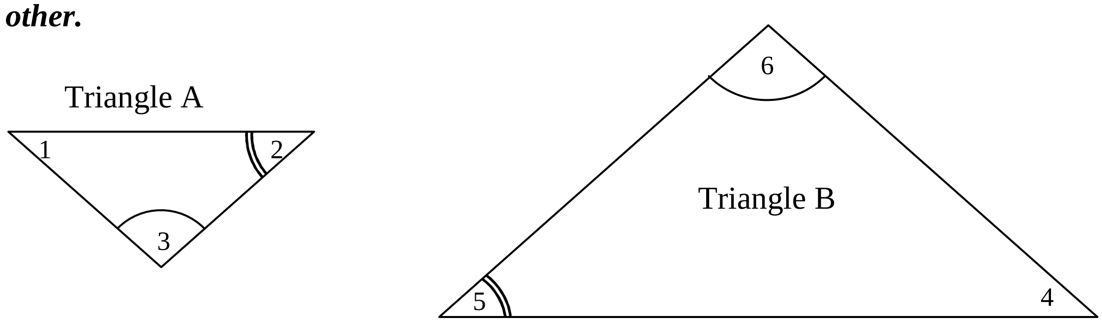
If a line divides two sides of a triangle proportionally, it is parallel to the third side.



If $\frac{WO}{WS} = \frac{WI}{WL}$, then we know that $\overleftrightarrow{IO} \parallel \overline{LS}$.

- **Proposition 70: Theorem 7.14**

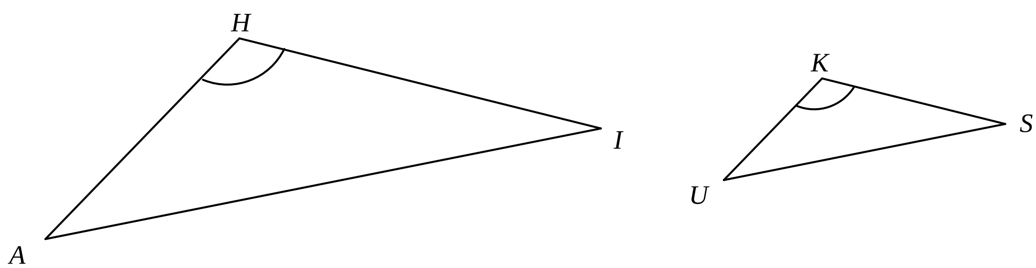
  Two triangles are similar if two angles of one are congruent, respectively, to two angles of the other.



Since $\angle 3 \cong \angle 6$ and $\angle 2 \cong \angle 5$, we conclude that triangle A is similar to triangle B.

- **Proposition 71: Theorem 7.15**

  Two triangles are similar if an angle of one is congruent to an angle of the other and the sides including these angles are proportional.

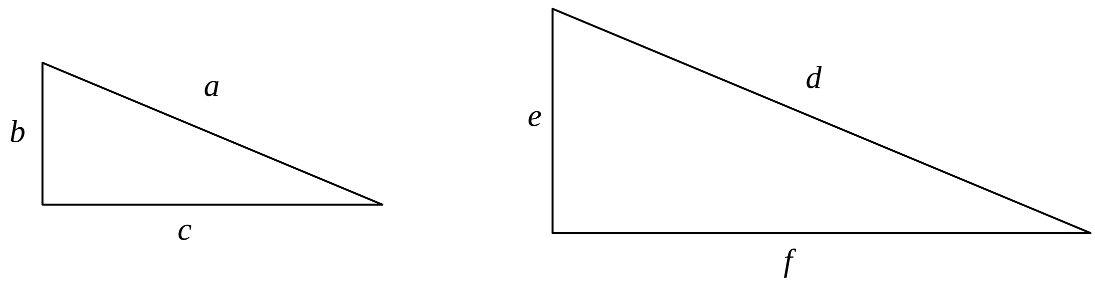


ΔHAI is similar to ΔKUS, if it is true that one of their angles is congruent $(\angle H \cong \angle K)$, and that the sides that make up the angle are proportional $\left(\frac{HA}{KU} = \frac{HI}{KS}\right)$.

Note: Same shape = similar Same shape and same size = congruent

- **Proposition 72: Theorem 7.16**

  Two triangles are similar if their corresponding sides are proportional.

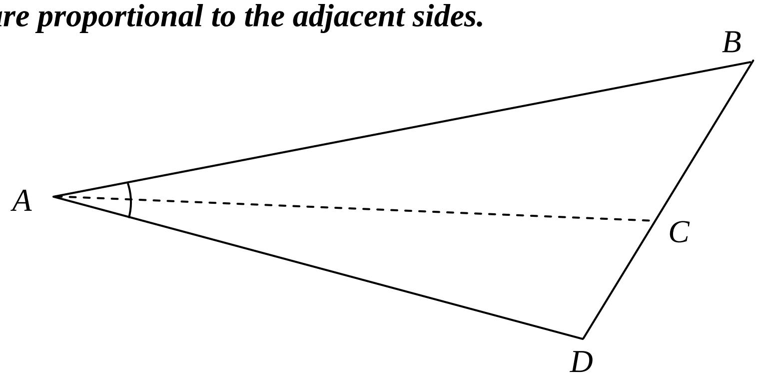


These triangles are similar if

$$\frac{a}{d} = \frac{b}{e} = \frac{c}{f}$$

- **Proposition 73: Theorem 7.17**

  The bisector of an interior angle of a triangle divides the opposite side into segments that are proportional to the adjacent sides.

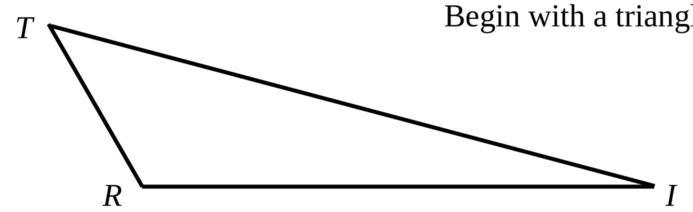


In triangle ΔABD, angle A has bisector $\overline{AC}$, cutting side $\overline{BD}$ in such a way that

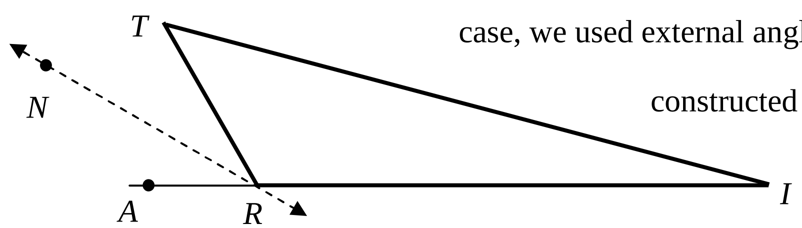
$$\frac{BC}{CD} = \frac{AB}{AD}.$$

- **Proposition 74: Theorem 7.18**

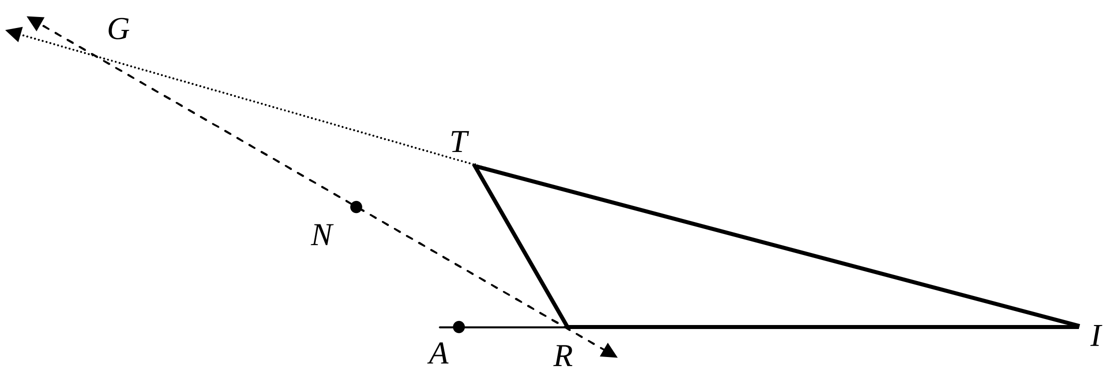
  If the bisector of an exterior angle of a triangle divides the opposite side of the triangle extended, it divides the opposite side externally into segments proportional to the adjacent sides.



Begin with a triangle, say ΔTRI



Find the bisector to an external angle. In this case, we used external angle $\angle TRA$, and constructed its bisector, $\overleftrightarrow{RN}$.

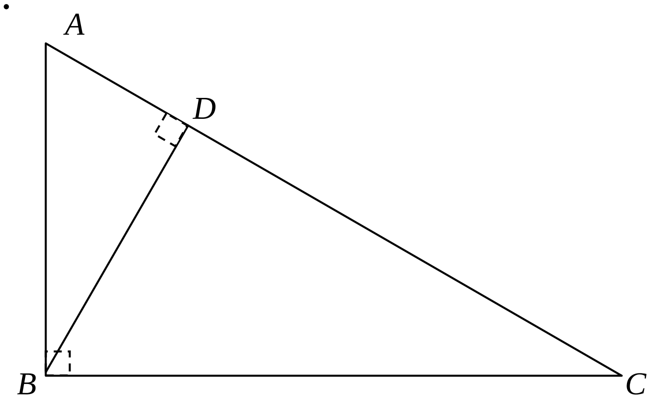


Then, if we extend both side $\overline{TI}$ and bisector $\overleftrightarrow{RN}$ until they cross at point *G*, the following is true:

$$\frac{RI}{RT} = \frac{GI}{GT}$$

- **Proposition 75: Theorem 7.19**

  In any right triangle, the perpendicular dropped from the vertex of the right angle to the hypotenuse divides the triangle into two triangles similar to the given triangle and to each other.

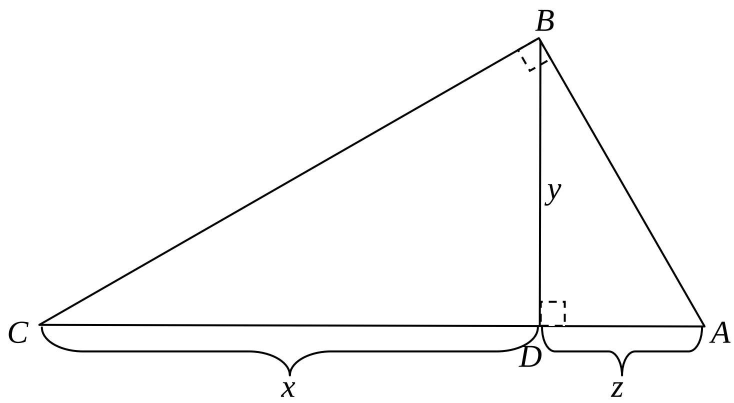


This proposition states that if $\overline{BD}$ is the altitude drawn from vertex of the right angle, then $\Delta ABC \sim \Delta ADB \sim \Delta BDC$.

Here's why it's true:

To prove that the triangles are similar, we need to show that they have two corresponding angles that are congruent (proposition 70). First we will prove $\Delta ABC \sim \Delta ADB$. Let's find some congruent angles in the two triangles. Since $\overline{BD}$ is the perpendicular line drawn from $\overline{AC}$, we know that $m\angle ADB = 90°$. We were also given $m\angle ABC = 90°$. The reflexive property allows us to say that $\angle A \cong \angle A$. Therefore, our corresponding angles are $\angle A \cong \angle A$ and $\angle ADB \cong \angle ABC$. Thus, $\Delta ABC \sim \Delta ADB$ by proposition 70. Now we will show that $\Delta ABC \sim \Delta BDC$. Since $\overline{BD}$ is the perpendicular line drawn from $\overline{AC}$, we know that $m\angle BDC = 90°$. The reflexive property allows us to say that $\angle C \cong \angle C$. Therefore, our corresponding angles are $\angle C \cong \angle C$ and $\angle BDC \cong \angle ABC$. Thus, $\Delta ABC \sim \Delta BDC$ by proposition 70. Lastly, we know that $\Delta ADB \sim \Delta BDC$, because they are both similar to the same triangle (ΔABC).

C1: ***In any right triangle, the perpendicular dropped from the vertex of the right angle to the hypotenuse, is the mean proportional between the segments of the hypotenuse.***

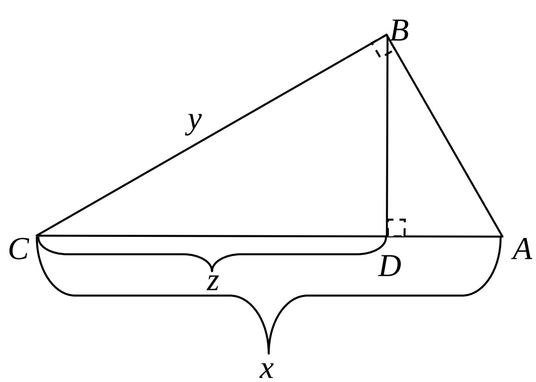


This corollary states that if $\overline{BD}$ is the altitude drawn from vertex with the right angle, then $x : y = y : z$. This can also be stated as $\frac{x}{y} = \frac{y}{z}$.

Remember that we stated $\Delta ADB \sim \Delta BDC$ in the last proposition. Since these two triangles are similar, we know that the lengths of the corresponding sides are proportional. Remember that $m\angle ADB = 90°$ and $m\angle BDC = 90°$, so the sides adjacent to those angles are the legs of a right triangle. Notice the two triangles share a common side $\left(\overline{BD}\right)$; however, this common side does not correspond to itself among the two triangles. In ΔADB, $\overline{BD}$ is the longer leg. In ΔBDC, $\overline{BD}$ is the shorter leg. The proportion, then, that shows the relationship between the lengths of the legs of the two triangles is:

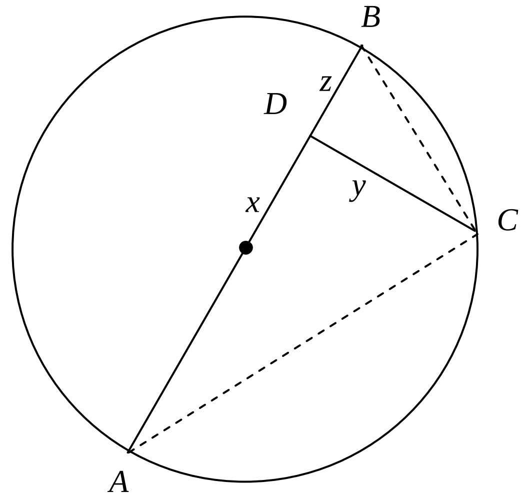
$$\frac{\text{longer leg}}{\text{shorter leg}} = \frac{x}{y} = \frac{y}{z}$$

C2: ***In any right triangle if the perpendicular is dropped from the vertex of the right angle to the hypotenuse, either side is the mean proportional between the hypotenuse and the segment of the hypotenuse next to it.***



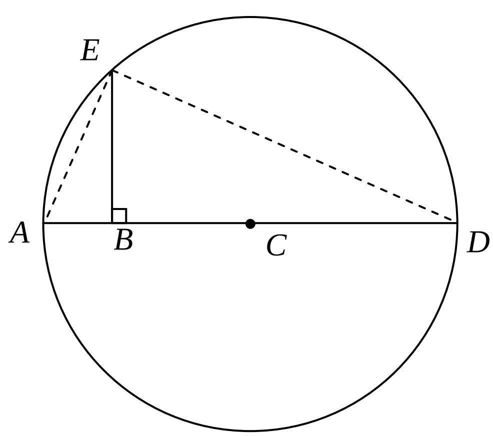
This corollary states that if $\overline{BD}$ is the altitude drawn from vertex with the right angle, then $x : y = y : z$. This can also be stated as $\frac{x}{y} = \frac{y}{z}$. This can be proven using proportions just as we did in the last corollary.

C3: ***The perpendicular to the diameter of a circle from any point on the circle is the mean proportional between the segments of the diameter.***



This can be proven using similar triangles just as in C1. Notice the dotted lines we drew to make ΔABC. The interesting thing is that $m\angle ACB = 90°$, because it is an inscribed angle of a half-circle, or a <u>semicircle</u>. Thus, ΔABC is a right triangle and we get the same proportion as we did for C1 of proposition 75.

C4: ***The perpendicular to the diameter of a circle from any point on the circle and the chord from that point to either extremity of the diameter is the mean proportional between the diameter and the segment of the diameter adjacent to that chord.***



$$\frac{EB}{ED} = \frac{BD}{AD}$$

$$\frac{EB}{EA} = \frac{AB}{AD}$$

The proof for this is identical to that of C2.

NOTES

End of Propositions